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A MODEL FOR RELATING WEAPONS SYSTEM
AVAILABILITY AND CONTINUOUS REVIEW
INVENTORY POLICIES FOR REPAIR PARTS

by

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ABSTRACT

A model is developed for relating continuous review inventory policies for repair parts to system availability. The system consists of S identical unit systems, each of which is a series of k-out-of-n structures. Unit system states are zero or one. An optimal cannibalization policy is assumed. Under this assumption the number of unit systems up is always the maximum possible for any given vector of backorders for the N part types in the system. The distribution of backorders under a (Q,r) policy with Poisson demands for each part type is used to derive expressions for system availability as functions of the $\underline{Q}$, $\underline{r}$ vectors. For simplicity it is assumed that order quantities are set by an operating level in terms of days of supply. A numerical technique is presented for finding the vector of reorder points (safety levels) which minimizes the expected cost of on hand inventory subject to one of two alternative availability constraints.

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LIST OF SYMBOLS AND ABBREVIATIONS

U_k	A random variable taking the values zero or one indicating the state of the kth unit system
Z	A random variable indicating the number of operational unit systems
Z^*	The value of Z after an optimal cannibalization
S	The total number of unit systems
$E(\cdot)$	The expected value operator
A_s	Required availability with respect to supply support, defined as a required ratio of $E(Z)$ to the total number of unit systems S, given that failed parts are replaced instantaneously if there are spares on hand
$P_{min}(k)$	Required assurance that at least k unit systems are up, given that failed parts are replaced instantaneously if there are spares on hand
λ_{ki}	The ith location where a part is installed on the kth unit system
Λ_k	$\{\lambda_{ki}\}_{i=1}^{n}$ = The set of all locations where parts are installed on the kth unit system
$\underset{\sim}{\Lambda}$	$\bigcup_{k=1}^{S} \Lambda_k$ = The set of nS locations on all unit systems where parts are installed
X_{ki}	A random variable taking the values zero or one indicating whether or not there is a working part of the proper type installed in the ith location on the kth unit system.
$\underline{X}_k$	A random vector whose components are X_{ki}, $i = 1,\dots,n$ indicating the states of all the locations requiring parts on the kth unit system
$\underline{X}$	$(\underline{X}_1,\dots,\underline{X}_S)$, random vector indicating the states of all locations requiring parts in the whole system
$\mathcal{X}_k$	The set of possible values of $\underline{X}_k$

$\mathcal{X}$	The set of all possible values of $\underline{X}$
ϕ	The empty set
FSN_j	the jth part type or the jth Federal Stock Number, $j = 1,\ldots,N$
Φ	$\{FSN_j\}_{j=1}^{N}$ = The set of different part types installed on each unit system
N	The number of different types of parts or number of different FSN's installed on each unit system, i.e., N = the size of Φ
$G(FSN_j)$	The subset of each Λ_k consisting of locations requiring part type j
a_j	The size of $G(FSN_j)$, i.e., the number of applications of part type j on each unit system
b_j	The minimum number of FSN_j which must be working on a unit system if it is to be operational
W_j	A random variable indicating the number of working parts of type j in the whole system
$\underline{W}$	A random vector, $(W_1,\ldots,W_N)$, where W_j is defined above
Y_j	A random variable indicating the number of backorders for part type j
$\underline{Y}$	A random vector, $(Y_{\cdot},\ldots,Y_N)$, where Y_j is determined above
$I_K(\underline{X})$	The indicator function of the set K at the point $\underline{X}$
t	Time
τ	Procurement lead time
Q	Order quantity
r	Reorder point
IP	Inventory position
$\psi_j(y_j;r_j)$	The probability mass function for backorders for part type j, given that the reorder point is r_j

$p(u;\rho\tau)$	The Poisson probability mass function at u, given that the mean is $\rho\tau$
$P(u;\rho\tau)$	The complementary cumulative Poisson distribution function at u, given that the mean is $\rho\tau$
$\hat{\rho}_j$	The failure rate for part type j
$_j$	$L_a S a_j \hat{\rho}_j$, the pooled failure rate for all parts of type j in the whole system, where L_a is defined below
L_a	Activity level multiplier used to account for varying levels of usage of the system
L_o	Operating level in terms of days or months of supply used to set order quantities

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I. INTRODUCTION

A. BACKGROUND

In the management of logistic support for weapons systems a significant problem in resource allocation is the determination of optimal inventory policies for repair parts. Inadequate stocks of repair parts result in low system availability. Stocks sufficient to insure with high probability that any part will be immediately available when needed my tie-up more resources than are justified.

An indication of the concern of the logistics manager with this problem is the following statement from the revised General Objective No. 1 for the Navy Supply Systems Command: "For technical material, optimum support is that which minimizes downtime of weapons systems due to lack of repair parts and components." [Ref. 1] In order to accomplish this objective, techniques are needed for relating repair parts inventory policies to weapons systems availability.

The mathematical models designed to deal with the impact of repair parts shortages have usually been formulated as a minimization of total cost, where the total cost includes ordering, holding, and backorder costs. In theory the backorder costs are a measure of the impact of shortages on system availability. A difficulty in practice is that backorder costs are hard to estimate. One technique for finding a value of the shortage cost is to impute it based on a required maximum probability P_{out} of being out of stock. See Deemer and Hoekstra [Ref. 2, p. 5] for a discussion of this technique. The required value of P_{out} is determined by management judgement. This judgement presumably includes some intuitive consideration of the effect of repair parts shortages on system availability.

The total cost minimization formulation does not, however, permit any direct correlation of inventory policies and system availability. Inventory policies computed using that formulation are determined for each part independent of all other parts. System availability is, however, a function of $(\underline{Q},\underline{r}) = (Q_1,r_1),\ldots,(Q_N,r_N)$. Therefore, if we wish to correlate inventory policies to system availability, we need a multi-item model rather than one which deals with each item one at a time.

There do exist a number of techniques for explicitly correlating system availability and single period inventory policies. One of these is the optimal redundancy approach discussed in Chapter 6 of Barlow and Prochan [Ref. 3]. The problem discussed there is one where at the beginning of the period there is a quantity n_j of each part type j on hand, and resupply is not available until the end of the period. The object is to determine the value of each n_j so that system availability is maximized subject to some cost constraint or, alternatively, to minimize the cost of achieving a required availability. The cost of the parts inventory may be interpreted as dollars, weight, volume or some other measure of the amount of a "resource" which is used up by the inventory.

B. GENERAL NATURE OF THE MODEL

The approach taken here is to apply the redundancy optimization ideas to a continuous review inventory situation. In particular the model addresses the problem of optimal parts inventory policies in support of a weapons system at the direct support echelon. The objective function to be minimized is the expected cost of on hand inventory. The constraint is a system availability requirement, which may be in one of two alternative forms. The first is that the expected number of unit systems up must be greater than or equal to a required fraction of the total number of unit systems. The second form of the constraint is that the probability that at least k unit systems are up must be equal to or greater than a required assurance level.

The model is an idealization of the repair parts supply support furnished by the direct support maintenance unit for the population of a particular major item in an operational Army unit. The system consists of S identical unit systems, which might be tanks, aircraft, howitzers, or some other set of equipments of the same make and model.

C. ASSUMPTIONS OF THE MODEL AND THEIR IMPLICATIONS

Before turning to the formal description of the model we shall discuss some of the important assumptions and their implications here.

1. <u>The Inventory System is Single Echelon</u>

To simplify the problem we assume that the supported military unit does not carry a stock of repair parts, but that parts are immediately available from the direct support maintenance unit if there is stock on hand at the time a demand occurs. The inventory system thus is single echelon.

2. <u>Procurement Lead Time is Constant</u>

The procurement lead time (order and ship time) is assumed to be constant for each part type, where part type indicates a particular Federal Stock Number. Lead times may, however, be different for different part types. We further assume that the source of supply is never out of stock. Thus, the model ignores variability of lead times and the possibility of reducing lead times by using a higher requisition priority.

3. <u>Unit System State Values are Zero or One</u>

The supported system consists of S identical unit systems. Each unit system is assumed to exist in one of exactly two states depending upon the states of the installed parts: either it is up, i.e., it is fully capable of operating satisfactorily; or it is down, i.e., totally ineffective. This assumption means that the possibility that the unit system may be partially effective is not considered. Actual equipment does exist in partially effective states. For example, if the 50-caliber machine gun on a tank is not working, the capability of the tank is reduced in some of its roles, but it is obviously not totally ineffective if all other components of the tank are working. The zero-one assumption is made for mathematical simplicity. The two state categorization is not too different from the equipment serviceability code (ESC) ratings currently used within the Army for determining the materiel readiness of Army units. Possible ESC ratings are green, meaning fully operational and capable of operating in combat for 60 days; amber, meaning fully operational and capable of operating in combat for 30 days; and red, meaning not fully operational. We have simply dropped the amber category.

4. Component Part State Values are Zero or One

Similar to the zero-one assumption for each unit system two possible states are assumed for each installed part: either it is working or it has failed. Thus the part is 100 per cent effective or totally ineffective.

5. The Unit System is a Series of k-out-of-n Structures

As mentioned in Section 3 above, the state of a unit system is determined by the states of its component parts. If there are a_j parts of type j initially installed on each unit system, it is assumed that there is a number b_j equal to or less than a_j such that at least b_j parts of type j must be working if the unit system is to be up. If the number of working type j parts is less than b_j, the unit system will be down. Further we assume that, if the number of working type j parts is equal to or greater than b_j for all j, then the unit system will be up. These assumptions may be summarized by saying that the unit system is a series of k-out-of-n structures, as defined by Birnbaum, Esary and Saunders [Ref. 4, p. 58]. Our b_j corresponds to these authors' k and our a_j to their n.

6. <u>The Number of Unit Systems Up is the Minimum of the Number Up with Respect to Each Part Type</u>

If Z is the number of unit systems up at some arbitrary time and Z_j is the number which would be up if all non-working parts other than type j were replaced by working parts, then Z is called the state of the system, and Z_j is called the state of the system with respect to part type j. We assume that at any time t the state of the system Z equals the minimum over j of the Z_j.

7. <u>An Optimal Cannibalization Policy is Observed</u>

Perhaps the most significant assumption of the model developed here is that a policy of optimal cannibalization is observed. By optimal we mean that given any vector $\underline{W} = (W_1, \ldots, W_N)$, where W_j denotes the number of working parts of type j, after the cannibalization operation the maximum possible number of unit systems will be up. The mathematical structure of the supported system is an adaptation of the structure of the cannibalization model of

Hirsch, Meisner and Boll [Ref. 5]. The reason for using this structure is that, given any vector $\underline{Y} = (Y_1,\ldots,Y_N)$, where Y_j denotes the number of backorders of part type j, there is a unique number systems up. This would not be the case if cannibalization were not allowed. Consider, for example, a system in which each unit system has two parts of type j installed, and both of these must be working if the unit system is to be up. Suppose there are two backorders for this part type. This means that there are two non-working parts of type j in the system. If one of these were on one unit system and the other were on a different unit system, then the number of unit systems down for this part type would be two. If, however, both were on one unit system, then only one unit system would be down for this part type. Under a policy of optimal cannibalization there would always be only one unit system down for this backorder situation. Since the probabilities for the number of systems up used in this model are computed based on the probability distributions of the Y_j, the cannibalization assumption is an essential feature of the model. The degree to which cannibalization is actually used to increase system availability in practice depends upon command policy and the practicality of taking parts from one unit system to make another operational. If the supported unit is spread-out geographically, it may not be feasible to cannibalize. Also the maintenance effort in removing and replacing parts under the cannibalization operation may overload the capability of the maintenance organization. Nevertheless, some cannibalization may be the best strategy for optimizing system availability when there are constraints on the quantities of repair parts the maintenance unit can carry.

8. The Demand Distribution for Each Part Type is Poisson

The demand distribution for each part type is assumed to be Poisson with mean equal to the product of the failure rate for the part type times the number of parts of that type in the system. The failure rate of each part type is assumed to be the same in all of its applications. The Poisson assumption implies that the mean time between failures of a given part type is exponentially distributed. It is true that the exponential distribution may be a poor fit for the mean time between failures where an individual part is installed and replaced with a new part of the same type immediately upon failure, particularly when the part is subject to wearout. Cox [Ref. 6, p. 77] indicates however, that the pooled output of a number of renewal processes tends to have the properties of a Poisson process as the number of renewal processes being pooled gets large. Consequently, the Poisson assumption may be a good approximation when the total number of parts of a given type in the system is large. Another implication of the assumption is that the distribution of demands is not affected by the number of unit systems down. This in turn implies that an inoperable unit system continues to generate part failures even after it has gone down. Errors introduced by this assumption will not be too great if the availability rate is reasonably high or if failure rates are estimated on demand histories over a period when the availability rate was about the same as the required availability rate for the period for which inventory policies are being computed.

9. Demand Rates are Linear with Activity Levels

The parameter in the demand distribution is assumed to be a time rate. As discussed by Soland [Ref. 7, p. 45], the natural parameter of the demand process may be miles driven, rounds fired, or some other measure of the usage or activity level of the system. The device used in the model to account for this fact is an "activity level" multiplier L_a for the pooled demand rates for each part type. Suppose, for example that rounds fired were the natural parameter for the demand process. Further suppose that a certain number of rounds were fired on the average by each unit system per month during the period when demand data were accumulated for estimating failure rates. Now suppose that during the period for which inventory policies are being computed that the programmed number of rounds per unit system per month is doubled. Then the L_a would be two, and the failure rate estimates for each part type would be doubled. Use of this device assumes that failure rates are linear with the level of usage. This assumption is not in general true, but any more realistic means of handling this problem would complicate the model.

For part types not subject to aging, this assumption is probably fairly good.

10. <u>Parts are Replaced Immediately upon Failure</u>

This model assumes that each part is replaced immediately upon failure with zero replacement time if a spare part is on hand at the time of failure. Also, if backorders exist for a particular part type, the failed part will be installed immediately upon receipt by the maintenance unit. Of course, these assumptions are not true in the "real world." To account for the fact that repair times are finite the following procedure could be used. First estimate the availability which we would expect to achieve if there never were any shortage of repair parts. Call this estimate the availability with respect to maintenance. Call the estimated availability predicted by this model the availability with respect to supply support. Then by taking the product of these two estimates we should have a fair estimate of the overall availability to be expected given the inventory policies and the repair capabilities of the maintenance unit. In any case, the reader should keep in mind that availability values computed with this model need to be corrected for down time due to non-zero repair times.

II. MATHEMATICAL MODEL

A. FORMAL DESCRIPTION OF THE SYSTEM STRUCTURE

The system consists of S identical unit systems. On each unit system k there are n loci where parts are installed. Let the λ_{ki} denote the ith locus on the kth unit system and let the set

$$\Lambda_k = \{\lambda_{ki}\}_{i=1}^{n}$$

denote the set of loci on the kth unit system. Let the set

$$\underset{\sim}{\Lambda} = \bigcup_{k=1}^{S} \Lambda_k$$

denote the set of all loci on the S unit systems. Let X_{ki} be a random variable indicating the state of the ith locus on the kth unit system, such that

$$X_{ki} = 1 \text{ if locus } \lambda_{ki} \text{ contains a working part of the type required in that locus,}$$

$$= 0 \text{ if locus } \lambda_{ki} \text{ fails to contain a working part of the required type.}$$

Now $\underline{X}_k = (X_{k1},\ldots,X_{kn})$ is a vector of zeros and ones which describes the state of the kth unit system. The possible values of $\underline{X}_k$ correspond to the vertices of the unit cube in Euclidan n-space. Let $\mathcal{X}_k$ denote the set of all possible values of $\underline{X}_k$. The size of $\mathcal{X}_k$ is 2^n. Let the random vector

$$\underline{X} = (\underline{X}_1,\ldots,\underline{X}_S)$$

be an nS-component vector indicating the state of the entire system. Let $\mathcal{X}$ denote the set of all possible values of $\underline{X}$. The size of $\mathcal{X}$ is 2^{nS}. Let the set

$$\Phi = \{FSN_j\}_{j=1}^{n}$$

represent the set of different part types installed in the n loci of each unit system. Further let $G(FSN_j)$ be the subset of Λ_k which has as elements all those loci requiring part type FSN_j. It is assumed that if locus $\lambda_{ki} \varepsilon G(FSN_j)$, then $X_{ki} = 1$ if and only if a working part of type FSN_j is installed in that locus. This assumption implies that different FSN's are not substitutes for one another. Let the size of $G(FSN_j) = a_j$, i.e., the number of applications of FSN_j on each unit system is a_j. Assume that for each j there exists a number b_j such that

1. $1 \leq b_j \leq a_j$;
2. If the number of working parts of type FSN_j installed on unit system k is equal to or greater than b_j, the unit system will not fail due to FSN_j;
3. If the number of working parts of type FSN_j installed on unit system k is less than b_j, the unit system will fail due to FSN_j; and
4. b_j is independent of b_i for all $i \neq j$.

Let the <u>state of the unit system with respect to FSN_j</u>, denoted U_{kj}, be defined as the state in which all loci not in the set $G(FSN_j)$ have working parts installed. In other words U_{kj} describes the state of the unit system if FSN_j were only part type subject to failure. If W_{kj} is a random variable indicating the number of working parts of type FSN_j installed on unit system k, then

$$U_{kj} = 1 \text{ if } W_{kj} \geq b_j,$$
$$= 0 \text{ otherwise.}$$

Now let the <u>state of the system with respect to FSN_j</u>, denoted Z_j, be defined as $\sum_{k=1}^{S} U_{kj}$. Z_j indicates the number of unit systems which would be operational for a given set of values of W_{kj}, $k = 1,\ldots,S$ if the only fallable part type were FSN_j.

For a given value of the system state vector $\underline{X}$ it may be possible to increase the value of Z by cannibalization. We assume that for any value of $\underline{X}$ cannibalization will be effected in such a way that the maximum value of Z for that $\underline{X}$ will be achieved. See Hirsch, Miesner and Boll [Ref. 5, p. 336-342] for a detailed description of the cannibalization operation for the type of structure being discussed here. Let Z^* denote the state of the system after an optimal cannibalization, and let Z_j^* denote the state of the system with respect to FSN_j under a policy of optimal cannibalization.

Note that Z_j^* may be considered to be a function of W_j the number of working parts of type j, since Z_j^* is the value of the number of operational unit systems after an optimal cannibalization when the only part failures are at those loci associated with part type j. It is assumed that $Z_j^*(W_j)$ is a monotone non-decreasing function of W_j. Let $K(W_j,z)$ denote the set of values of the number of working parts W_j such that the value of Z_j^* is at least as great as z, i.e.,

$$K(W_j,z) = \{W_j: \ 0 \leq W_j \leq Sa_j \text{ and } Z_j^*(W_j) \geq z\}.$$

Note that

$$K(W_j,0) \subset K(W_j,1) \subset \cdots \subset K(W_j,S) \subset K(W_j,S+1), \quad j=1,\ldots,N;$$

and since max $Z_j^* = S$,

$$K(W_j,S+1) = \phi \ .$$

The largest value of z such that $K(W_j,z) \neq \phi$ is S. Further note that the minimum value of W_j such that Z $Z_j^* = z$ is zb_j since b_j is the minimum number of part type j needed on each unit system if that unit system is to be operational, in symbols

$$zb_j = \min \{w_j: \ w_j \varepsilon K(W_j,z)\} \ .$$

Let $K\varepsilon\chi$ be an arbitrary set where χ is the set of all system state vectors $\underline{X}$. The indicator function I_K is defined for all K and $\underline{X}$ as follows

$$\begin{aligned} I_K(\underline{X}) &= 1, \text{ if } \underline{X}\varepsilon K, \\ &= 0, \text{ if } \underline{X}\notin K \ . \end{aligned}$$

Now let us use the symbol $\{w_j \geq zb_j\}$ to denote the set of all $\underline{X}$ such that $w_j \geq zb_j$, i.e., $\{\underline{X}\varepsilon\mathcal{X}: w_j(\underline{X}) \geq zb_j\}$. Hirsch, Meisner and Boll show (using somewhat different notation) that if $Z^* = \min_j Z_j^*$, then

$$Z^* = \sum_{k=1}^{S} I_{\bigcap_{j=1}^{N}\{w_j \geq zb_j\}} = \sum_{k=1}^{S} \prod_{j=1}^{N} I_{\{w_j \geq zb_j\}},$$

and the authors call this the "representation theorem" [Ref. 5, p. 349]. If $S = 1$, i.e., we have only one unit system then $Z^* = U$, and $U = 1$ if and only if each one of the indicators $I_{\{w_j=b_j\}}$ is equal to 1. The structure of the unit system in this model is thus a coherent structure in the definition of that term given by Birnbaum, Esary and Saunders [Ref. 4, p. 61]. Further each unit system may be considered to be a k-out-of-n structure with respect to each part type j, where a k-out-of-n structure is one which has n parts of a single type and is operational if and only if at least k out of the n parts are working. Also, we can consider each unit system to be a series system composed of N k-out-of-n structures, where the k's are the b_j's and the n's are the a_j's.

B. CORRELATION OF INVENTORY POLICIES AND SYSTEM AVAILABILITY

Consider now the effect of inventory policies on the expected value of the number of operational unit systems E(Z). If there were an infinite number of repair parts of each type available, the value of $\underline{X}$ would always be

$\underline{1} = (1,1,\ldots,1)$, since it is assumed that each part is replaced immediately upon failure as long as spares are available, and the state of the system Z would always be S. There are in fact various constraints on the ability of the support unit to carry inventories of parts. The result is that from time to time demands will occur for which no replacement part is immediately available. For any state vector $\underline{X}$, denote the state vector after an optimal cannibalization as $\underline{X}^*$ and the value of Z after cannibalization as Z^*. Then

$$\underline{X}^*(t) = (X^*_{1,1}(t),\ldots,X^*_{S,n}(t)),\ t \geq 0,$$

is a stochastic process where for each fixed t, X_{ki} is a random variable taking the values zero or one. Now for each subset K of $\mathcal{X}$, the set of possible values of $\underline{X}$, let $P_t(K)$ denote the probability that $\underline{X}^*(t)$ is an element of the set K, i.e.,

$$P_t(k) = \Pr\{\underline{X}^*(t) \varepsilon K\},\quad K \subset \mathcal{X}\ .$$

As shown by Hirsch, Meisner and Boll the probability distribution defined above concentrates all of its mass on the set of maximum points $M_{\underline{X}}$, which are the possible vectors $\underline{X}^*$, the state vector after an optimal cannibalization. Any vector $\underline{X}$ in which the number of working parts of each type is given by the vector $\underline{W}$ can be transformed into another vector $\underline{X}'$ which has the same value of $\underline{W}$. Thus, since an optimal cannibalization yields the maximum value of the system structure function Z for a given state vector $\underline{X}$, there is a unique value of the state of the system after an optimal cannibalization Z^* for each value of the vector $\underline{W} = (W_1,\ldots,W_N)$. The set of all $\underline{X}$ corresponding to a given $\underline{W}$ vector is called an equivalence class, and the set $M_{\underline{X}}$ is a subset of this equivalence class where Z as a function of $\underline{X}$ takes its maximum values over the equivalence class. We denote each $\underline{X}$ in $M_{\underline{X}}$ as $\underline{X}^*$. To describe the variation in time of the system state function Z, set

$$Z^*(t) = Z(\underline{X}^*(t)),\ t \geq 0,$$

and we note that the probability distribution of $Z^*(t)$ is given by

$$\begin{aligned}\Pr\{Z^*(t) \geq z\} &= \Pr\{Z^*(x^*(t)) \geq z\}\\ &= P_t\{\underline{X}\colon Z^*(\underline{X}) \geq z\}\ .\end{aligned}$$

Let $W_j(t)$ represent the number of working spares of type j at time t. Under the assumption that $Z^* = \min_j Z_j^*$, we have immediately from the representation theorem that

$$Z^*(t) = \sum_{k=1}^{S} I_{\bigcap_{j=1}^{N}\{W_j(t)\geq kb_j\}} = \sum_{k=1}^{S} \prod_{j=1}^{N} I_{\{W_j(t)\geq kb_j\}}.$$

Assume that

1. For each index $j = 1,\ldots,N$, the loci in which parts of type FSN_j occur are indistinguishable in their

effects on the lifetimes of the parts installed in them, i.e., the failure rate at each instant of a given part type FSN_j does not depend on the locus in $G(FSN_j)$ in which the part is installed, nor on the particular sequence of loci through which it has passed.

2. Parts operate independently, i.e., the lifetime of a given part is not related to the lifetimes of any other parts. These assumptions make it reasonable to postulate that the joint distribution of $(W_1(t),\ldots,W_N(t))$ does not depend upon the particular cannibalizations involved in the process $\{Z^*(t),\ t \geq 0$ and that the N stochastic processes

$$\{W_1(t),\ t \geq 0\},\ \ldots,\ \{W_N(t),\ t \geq 0\}$$

are mutually independent. Thus the expected number of unit systems up under optimal cannibalization at time t, $E(Z^*(t))$, is given by

$$E(Z^*(t)) = \sum_{k=1}^{S} \prod_{j=1}^{N} E[I_{\{W_j(t) \geq kb_j\}}]$$

$$= \sum_{k=1}^{S} \prod_{j=1}^{N} \Pr\{W_j(t) \geq kb_j\}.$$

Further, the probability that $Z^*(t)$ equals at least z is given by

$$\Pr\{Z^*(t) \geq z\} = \prod_{j=1}^{N} \Pr\{W_j(t) \geq zb_j\}.$$

Now let $Y_j(t)$ be a random variable denoting the number of backorders for part type FSN_j at time t. Under the assumption that every failed part is replaced immediately by a working part from the inventory of spares, we see that

$$W_j(t) = Sa_j - Y_j(t)$$

and

$$\{W_j(t) \geq kb_j\} = \{Sa_j - Y_j(t) \geq kb_j\} = \{Y_j(t) \leq Sa_j - kb_j\}.$$

Hence

$$\Pr\{Z^*(t) \geq k\} = \prod_{j=1}^{N} \Pr\{Y_j(t) \geq Sa_j - kb_j\}$$

and

$$E(Z^*(t)) = \sum_{k=1}^{S} \sum_{j=1}^{N} \Pr\{Y_j(t) \geq Sa_j - kb_j\}.$$

We assume that the inventory system is continuous review with a (Q_j, r_j) policy for each FSN, where Q_j denotes order quantity and r_j denotes reorder point. We also assume the lead time demand is Poisson distributed. Hadley and Whitin [Ref. 8, p. 184] show that under a (Q,r) policy when the lead time demand is Poisson with parameter $\rho\tau$ that the probability that there are y backorders at an arbitrary time t under steady conditions may be derived as follows. The inventory position, IP, is defined as the stock on hand plus stock on order minus backorders. The IP varies between $r+1$ and $r+Q$, where r is defined in terms of the IP. Further, Hadley and Whitin show that the probability that the IP is in any state $r+j$, $j=1,\ldots,Q$ is $1/Q$. Thus if the inventory system is in state $r+j$ at time $t-\tau$, the probability of y backorders at time t equals the probability that $y+r+j$ demands occur during the period $(t-\tau)$ to t for $y>0$. The probability that $y=0$ is the probability that the demand during $(t-\tau,t)$ is less than or equal to $r+j$. Hadley and Whitin do not make this distinction for $y=0$; consequently, the formula they derive is not valid for $y=0$. When 0 is substituted for y in the expression they derive the resulting value is the probability that $y=0$ and on hand inventory = 0. Terms for the probability that $y=0$ and on hand inventory is greater than zero are left out of their expression. The expression Hadley and Whitin derive is

$$\psi = (1/Q) \sum_{j=1}^{Q} p(y+r+j;\rho\tau) = (1/Q) \sum_{u=y+r+1}^{y+r+Q} p(u;\rho\tau)$$

$$= (1/Q)[P(y+r+1;\rho\tau) - P(y+r+Q+1;\rho\tau)],$$

where $p(u;\rho\tau)$ is the Poisson probability mass function with parameter $\rho\tau$ at the point u for $u = 0,1,2,\ldots$; and $P(u;\rho\tau) = \sum_{v=u}^{\infty} p(v;\rho\tau)$ is the Poisson complementary cummulative distribution function. The above expression is valid y greater than zero. For $y = 0$,

$$\psi(0) = (1/Q) \sum_{j=1}^{Q} \sum_{u=0}^{r+j} p(u;\rho\tau)$$

$$= (1/Q) \sum_{j=1}^{Q} [1 - P(r+j+1;\rho\tau)]$$

$$= 1 - (1/Q) \sum_{j=1}^{Q} P(r+j+1;\rho\tau).$$

We assume that the pooled demand rate ρ_j for all parts of type j is given by

$$\rho_j = L_a S a_j \hat{\rho}_j ,$$

where L_a is the activity level multiplier discussed in Section I, S is the number of unit systems, a_j is the number of applications of part type j on each unit system, and $\hat{\rho}_j$ is the failure rate for part type j when L_a is one.

Let τ_j be the procurement lead time for part type j. Demand is assumed to be Poisson distributed with parameter $\rho_j\tau_j$. Thus in terms of $\psi_j(y_j)$,

$$\Pr\{Z_j^*(t) \geq k\} = \sum_{y_j=0}^{Sa_j-kb_j} \psi_j(y_j).$$

Also since $Z^*(t) = \min_j Z_j^*(t)$ and the y_j are assumed to be independent, the probability that $Z^*(t)$ is equal to or greater than k is the probability that each $Z_j^*(t)$ is equal to or greater than k, i.e.,

$$\Pr\{Z^*(t) \geq k\} = \prod_{j=1}^{N} \sum_{y_j=0}^{Sa_j-kb_j} \psi_j(y_j).$$

Now the expected value of a non-negative integer random variable is the sum from 1 to ∞ of the complementary cumulative probability function; thus

$$E[Z^*(t)] = \sum_{k=1}^{\infty} \{\Pr Z^*(t) \geq k\} = \sum_{k=1}^{S} \Pr\{Z^*(t) \geq k\},$$

since the probability that $Z^*(t)$ is greater than S is zero. Therefore, the expected value of the number of operational systems in terms of the probability mass functions of the backorders for each part type is:

$$E[Z^*(t)] = \sum_{k=1}^{S} \prod_{j=1}^{N} \sum_{y_j=0}^{Sa_j-kb_j} \psi_j(y_j),$$

where:

$$\psi_j(y_j) = (1/Q)[P(y_j+r_j+1) - P(y_j+r_j+Q_j+1)], \text{ for } y_j>0,$$

and

$$\psi_j(y_j) = 1 - (1/Q) \sum_{m=1}^{\bar{Q}} P(r_j+m+1), \text{ for } y_j = 0.$$

The problem of optimum inventory policies may be formulated as the minimization of the expected cost of on hand inventory subject to one of the following constraints: first, that the expected number of operating systems $E(Z)$ must be equal to or greater than some required fraction of the total number, or, second, that the probability that at least k of the unit systems are operational must be equal to or greater than a minimum assurance level. The expressions used to relate inventory policies and these constraints are those developed above for $E(Z)$ and $Pr\{Z \geq k\}$. Note that since the expressions refer to "steady state" conditions, and since optimal cannibalization is always performed, the symbol Z will be used henceforth in lieu of $Z^*(t)$ and Z will indicate the "steady state" random variable $Z^*(t)$.

It turns out that for part types with low demand rates the optimal policy under the formulation of the problem indicated above is not to stock these low demand items at the direct support level, but to order them from the source of supply as demands occur. If this is the case, r will be set to -1 and Q to 1. This means that an order is placed as soon as a failure occurs. In this situation the distribution of Y_j reduces to the Poisson distribution with parameter $\rho_j\tau_j$, since the inventory position is always equal to zero, and there is, therefore, only one state for the inventory position. Hence the probability that there are exactly y_j backorders at any arbitrary time is $p(y_j;\rho_j\tau_j)$.

It is interesting to consider what would happen if $\rho_j\tau_j$ were small for all N part types, and no stock were carried at the direct support level. Assume that each part type were always available from the source of supply with a procurement lead time of τ_j, a constant, which might be different for different part types. Now, if no parts are carried at direct support level, but are ordered only when demands occur, i.e., $r_j = -1$, $Q_j = 1$ for all j, then

$$\Pr\{Z \geq k\} = \prod_{j=1}^{S} \sum_{y_j=0}^{Sa_j-kb_j} p(y_j;\rho_j\tau_j),$$

and

$$E(Z) = \sum_{k=1}^{S} \prod_{j=1}^{N} \sum_{y_j=0}^{Sa_j-kb_j} p(y_j;\rho_j\tau_j).$$

The values of the above functions can be used to estimate the lower bounds for $E(Z)$ and $\Pr\{Z \geq k\}$, when demand rates and procurement lead times are ρ_j, τ_j, $j=1,\ldots,N$.

III. OPTIMIZATION TECHNIQUE AND NUMERICAL ANALYSIS

A. THE COST FUNCTION

The objective function to be minimized is a cost function of the following form:

$$E(C) = \sum_{j=1}^{N} C_j D(Q_j, r_j),$$

where

$E(C)$ = Expected cost of on hand inventory in dollars, cubic feet, pounds or some other measure of a resource in short supply,

C_j = The unit cost of part type j,

$D(Q_j, r_j)$ = The expected on hand inventory of part type j, given order quantity Q_j and reorder point r_j.

If $E(C)$ is in dollars, the above function gives the expected amount of funds tied up in repair parts inventory for the system at the direct support echelon. If $E(C)$ were in pounds the function might represent the expected amount of load carrying capacity of the maintenance used up by the repair parts for the system in question.

The value of $D(Q_j, r_j)$ is given by

$$D(Q_j, r_j) = (Q_j+1)/2 + r_j - \rho_j \tau_j + B(Q_j, r_j),$$

where

$$B(Q_j, r_j) = \sum_{y_j=0}^{\infty} y_j \psi_j (y_j; Q_j, r_j)$$

is the expected number of backorders, given the inventory policy (Q_j, r_j). Hadley and Whitin [Ref. 8, p. 184] show that

$$B(Q,r) = (1/Q) \sum_{y=0}^{\infty} y[P(y+r+1;\rho\tau) - P(y+r+Q+1;\rho\tau)].$$

B. OPTIMIZATION TECHNIQUE

1. Minimizing Cost Subject to a Required Expected Number of Unit Systems Up

Suppose it is required that the expected number of unit systems up must be equal to or greater than $A_s S$, where A_s is the required availability with respect to supply support, $0 \leq A_s \leq 1$, and S is the number of unit systems. Further, suppose it is desired to minimize the cost of expected on hand inventory needed to achieve $E(Z) \geq A_s S$. The problem may then be stated as

$$\underset{\substack{(r_1,\ldots,r_N)\\(Q_1,\ldots,Q_N)}}{\text{Minimize}} \quad \sum_{j=1}^{N} C_j D(Q_j, r_j)$$

subject to $E(Z) \geq A_s S$.

Finding the optimal solution to this problem for (Q_j, r_j), $j = 1,\ldots,N$ is a formidable task due to the complexity of the cost function and the function for the expected value of Z in terms of the Q_j and r_j. To simplify the problem we have chosen to set the values of Q_j by an operating level of supply in terms of days or months of supply. Thus we set

$$Q_j = [\rho_j L_o + .5], \text{ if } [\rho_j L_o + .5] > 0,$$

$$= 1, \text{ otherwise,}$$

where: $[u]$ = The greatest integer equal to or less than u,

L_o = The operating level of supply, e.g., one month of supply.

The problem now becomes: given the values of Q_j determined as indicated above, find the vector $(r_1, \ldots, r_N)$ which minimizes the cost function subject to the required level of availability. We shall restrict the r_j to be ≥ -1. After computing the Q_j values, the next step in the procedure is to compute a table of values of $\psi_j(y_j;\ r_j = -1)$ for all j. This table is useful in that the values of $\psi_j(y_j;\ r_j)$ may be found for other values of r_j as follows:

$$\psi_j(0;\ r_j) = \sum_{u=0}^{r_j+1} \psi_j(u;\ -1), \text{ for } y_j = 0,$$

$$\psi_j(y_j;\ r_j) = \psi_j(y_j+r_j+1;\ -1), \text{ for } y_j > 0.$$

The number of values needed for each ψ_j depend upon the demand rate and the accuracy desired in the computations. After the table of ψ values is computed the next step is to find a set $(r_1^*, \ldots, r_N^*)$ of r_j such that $E(Z_j)$, the expected value of Z with respect to part type j, is equal to or greater than $A_s S$, and such that $Pr(Z_j=S;r_j)$ is concave for $r_j \geq r_j^*$. Each r_j must be large enough to insure $E(Z_j) \geq A_s S$ since $E(Z_j)$ gives the expected number of operating systems when every other type except j is always in supply. $E(Z)$ can not be greater than $E(Z_j)$ because it is assumed that at any arbitrary time $Z = \min_j Z_j$. Now $E(Z_j)$ is given by

$$E(Z_j) = \sum_{k=1}^{S} \sum_{y_j=0}^{Sa_j-kb_j} \psi_j(y_j;\, r_j).$$

Using this expression find the minimum r_j such that $E(Z_j) \geq A_s S$. (See Appendix A for a discussion of the need to require each $Pr(Z_j \geq k;\, r_j)$ to be concave for $r_j \geq r_j^*$.) After the set of initial values of the r_j is found, a marginal analysis technique similar to the approach of Karr and Geisler [Ref. 9] and to one form of optimal redundancy algorithm of Barlow and Prochan [Ref. 3, p. 166] is used to find at each iteration the part type which yields the greatest increase in $E(Z)$ for the increase in the cost of the expected on hand inventory for that part type. On the first iteration the ratio

$$\Delta_j = \frac{E(Z;r_j+1) - E(Z;r_j)}{C_j[D_j(r_j+1) - D_j(r_j)]}$$

is computed for each part type, where

$$E(Z;r_j+1) = \sum_{k=1}^{S} Pr(Z \geq k) \frac{Pr(Z_j \geq k;r_j+1)}{Pr(Z_j \geq k;\, r_j)}.$$

The maximum over j of Δ_j is determined and the r_j corresponding to that maximum is increased by one. $E(Z)$ is set equal to $E(Z;r_j+1)$ corresponding to this j. If the new value of $E(Z)$ is equal to or greater than $A_s S$, the procedure stops; if not, a new set of Δ_j is computed and the r_j corresponding to the maximum over j of the new Δ_j is increased by one. The procedure continues in this fashion until the value of $E(Z)$ is equal to or greater than $A_s S$.

2. Minimizing Cost Subject to a Required Probability that at Least k Unit Systems are Up

Suppose that it is required that $P\{z \geq k\} \geq P_{min}(k)$, and that it is desired to minimize the cost of the expected on hand inventory needed to achieve this probability. The general procedure for finding the optimum vector $(r_1,\ldots,r_N)$ is the same as the $E(Z) \geq A_s S$ form of the constraint with the following expections:

a. The initial r_j values $(r_1^*,\ldots,r_N^*)$ are the minimum values such that $Pr(Z_j \geq k) \geq P_{min}(k)$ and $Pr(Z_j \geq k_j r_j)$ is concave in r_j for $r_j \geq r_j^*$.

b. At each iteration Δ_j is computed as

$$\Delta_j = \frac{Pr\{Z \geq k;r_j+1\} - Pr\{Z \geq k;\ r_j\}}{C_j[D(r_j+1) - D(r_j)]} .$$

As in the first form of the constraint the r_j associated with the maximum Δ_j is increased by one at each iteration until $Pr(Z \geq k) \geq P_{min}(k)$. Since $Pr\{Z \geq k\}$ is just one term of the sum over k in the expression for $E(Z)$ the computational effort at each iteration is much reduced for this form of the constraint, particularaly if the number of unit systems S is large.

C. NUMERICAL ANALYSIS AND DEVELOPMENT OF FORTRAN PROGRAMS

Two FORTRAN programs, one for each form of the constraint have been developed and run for several sample problems on the Naval Postgraduate School's IBM 360 computer. The programs are very similar, and both use identical versions of the following subprograms: PSITAB, which computes a table of values of $\psi_j(y;r_j = -1)$ for $j = 1,\ldots,N$; PRZJK, a function subprogram which computes $Pr\{Z_j \geq k;r_j\}$; EBO, a function which computes the expected number of backorders for part type j give that the reorder point is r_j; PPT, a function which computes individual terms of the Poisson probability mass function; and PCUMT, which computes complementary cumulative terms of the Poisson distribution.

With the IBM 360 it was found necessary to do the arithmetic in double precision because the single precision round-off errors for repeated multiplications were excessive. The table of $\psi_j(y;r_j = -1)$ values was, however, stored as a single precision array to conserve storage space. When values from this table were needed they were converted to double precision with the standard function DBLE.

Subroutine PSITAB stores the computed values of $\psi_j(y;r_j = -1)$ in an array called PI(J,K), where

$$PI(J,K) = \Psi_J(k-1; r_J = -1).$$

The array size is (N,100), where N is the number of different part types; however, for each J only values of PI(J,K) for K less than or equal to KMAX(J) are computed, where

$$KMAX(J) = \min \begin{Bmatrix} 100, & \text{largest K such that} \\ & PI(J,K) < 10^{-12} \end{Bmatrix}.$$

PI(J,K) for K greater than KMAX(J) are set to zero. For part types with low demand rates the number of values of PI(J,K) computed is much less than 100. The array size used in the programs permits handling demand rates up to about 50 without significant truncation of the distribution of Y_j.

1. Program EZMIN

Program EZMIN computes the optimum $\underline{r}$ vector subject to a required minimum expected value of the number unit systems up. The main program reads the input data, writes the system parameters and the required availability, and calls subroutines PSITAB, INITAL, and OPTIMR. Subroutine PSITAB computes the PI(J,K) array as discussed above. Subroutines INITAL computes an initial set of r_j values, $r_1^*,\ldots,r_N^*$, each of which satisfies the following conditions: $Pr(Z_j = S; r_j)$ a concave function of r_j for $r_j \geq r_j^*$;

$$E(Z_j) = \sum_{k=1}^{S} Pr(Z_j \geq k; r_j^*) \geq A_S S.$$

Subroutine OPTIMR uses the marginal analysis technique previously discussed to find the optimal $\underline{r}$ vector.

2. Program PKSMIN

Program PKSMIN computes the optimum $\underline{r}$ vector subject to a required assurance that at least k out of S unit systems are operational. This program differs from program EZMIN only in the following respects. PKSMIN Main calls subroutine INTLZ instead of INITAL and subroutine OPTMZ instead of OPTIMR. Subroutine INTLZ finds an initial set of r_j values $r_1^*,\ldots,r_N^*$ such that $Pr(Z_j \geq k; r_j)$ is concave for $r_j \geq r_s^*$ and such that $Pr(Z_j \geq k; r_j^*) \geq P_{min}$. Subroutine OPTMZ computes the optimum $\underline{r}$ vector subject to the required assurance that Z be equal to or greater than k.

3. Flow Charts

Flow charts for program EZMIN are presented in Appendix B. Flow charts for program PKSMIN are presented in Appendix C.

IV. EXAMPLE PROBLEMS AND DISCUSSION OF RESULTS

A. DESCRIPTION OF THE EXAMPLE SYSTEM

The example system consists of 50 106-mm recoilles rifles, model M40A1. Several factors made this system a convenient example. First, the total number of repair parts is small enough so that the FORTRAN programs developed for solution of the two alternate formulations of the problem can be used without modification on the IBM 360 G-level computer available at the Naval Postgraduate School. Second, Department of the Army Technical Manual 9-1015-221-35 [Ref. 11] contains data from which failure rates may be estimated. Third, actual demand data for a six month period for 62 weapons used in training heavy weapons infantrymen at Fort Ord, California, were made available to the author by the Fort Ord Post Maintenance Section. Finally, price data were available from the microfilm Army Master Data File Selected Management Data File [Ref. 11].

The parts list in TM 9-1050-221-35 contains 287 different Federal Stock Numbers (FSN). Of these only the 159 FSN which were indicated as being combat essential were used for the sample problems. By limiting consideration to only combat essential parts we insure that some of the assumptions of the model are more nearly satisfied than if all part types were considered. For example, the model assumes that for each part type j there is a positive number b_j which is the minimum number of working parts of type j needed for the unit system to be capable of performing its function. A part would not be combat essential if b_j were zero. Thus the assumption that the unit system is a series of k-out-of-n structures and that the possible states are zero or one is more nearly true if the unit system consists only of combat essential parts.

The Federal Stock Number, the number of applications per unit system, and data for estimating failure rates were obtained from columns 2, 4, and 6, respectively of Section 2, Appendix B, Ref. 10.

B. SAMPLE PROBLEM 1 - REQUIRED EXPECTED NUMBER OF UNIT SYSTEMS UP CONSTRAINT

The first sample problem was computed for the required expected number of unit systems up form of the constraint. Table I lists a summary of the input data and results for this problem.

TABLE I

SAMPLE PROBLEM 1 - SUMMARY OF INPUT DATA AND COMPUTED RESULTS

INPUT DATA	NUMBER OF UNIT SYSTEMS	50
	NUMBER OF PART TYPES	159
	REQUIRED AVAILABILITY	0.95
	OPERATING LEVEL	1.0
	ACTIVITY LEVEL	1.0
COMPUTED RESULTS	EXPECTED NUMBER OF OPERATIONAL UNIT SYSTEMS	47.58
	EXPECTED COST OF ON HAND INVENTORY	$418.04
	NUMBER OF ITERATIONS OF MARGINAL ANALYSIS PROCEDURE	295

The above data show that for the system consisting of 50 weapons with 159 part types and a required availability of 0.95 program EZMIN computed an $\underline{r}$ vector with an associated expected cost of on hand inventory of $418.04. The computed value of $E(Z)$ is 47.58, which indicates that the marginal analysis technique overshot the required expected number of systems up by .08, since the required expected number was 50 x 0.95 = 47.50.

Table II lists the input parameters and computed results for each FSN. Parts are listed in Federal Item Identification Number (FIIN) sequence, because they are listed this way on the Army Master Data File [Ref. 11]. The FIIN is the last seven digits of the Federal Stock Number. The data cards were sorted into this sequence to facilitate determination of unit prices. The columns of Table II contain the following: j, the sequence number; FSN, the Federal Stock Number; $A = a_j$, the number of applications of part type j on each unit system; $B = b_j$, the minimum number of part type j needed for the unit system to be up; $C = C_j$, the unit cost of part type j; RHOHAT = $\hat{\rho}_j$, the failure rate for part type j; RHO = $\rho_j = Sa_j\hat{\rho}_j$, the pooled demand rate for all parts of type j in the system; TAU = τ_j, the procurement lead time for part type j; $Q = Q_j$, the order quantity; $R = r_j$, the reorder point; EB = $B(Q_j,r_j)$, the expected backorders for part type j; and EOH = $D(Q_j,r_j)$, the expected on hand inventory for part type j.

TABLE II

SAMPLE PROBLEM 1 - REQUIRED EXPECTED NUMBER UNIT SYSTEMS UP CONSTRAINT - INPUT PARMETERS AND COMPUTED VALUES OF ORDER QUANTITIES, REORDER POINTS, EXPECTED BACKORDERS AND EXPECTED ON HAND QUANTITIES FOR ALL PART TYPES

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
1	5305 013 9135	1	1	0.05	0.01400	0.700	1.00	1	2	0.0066	2.3066
2	5305 013 9818	6	6	0.05	0.01500	4.500	1.00	4	4	0.2913	[illegible].2913
3	1005 047 3861	1	1	16.70	0.00800	0.400	1.00	1	-1	0.4000	0.0
4	1015 049 5241	2	2	0.42	0.00700	0.700	1.00	1	0	0.1966	0.4966
5	1015 049 5244	3	3	12.47	0.00300	0.450	1.00	1	-1	0.4500	0.0
6	5315 052 6697	1	1	0.65	0.02800	1.400	1.00	1	3	0.0182	2.6182
7	5315 058 6047	1	1	0.01	0.07200	3.600	1.00	4	10	0.0002	8.9002
8	5315 058 6065	1	1	0.01	0.07200	3.600	1.00	4	10	0.0002	8.9002
9	5315 058 6079	2	2	0.39	0.05600	5.600	1.00	6	8	0.0317	5.9317
10	5315 058 6093	1	1	0.01	0.01600	0.800	1.00	1	4	0.0002	4.2002
11	5315 058 6094	1	1	0.26	0.07200	3.600	1.00	4	7	0.0064	5.9064
12	5315 059 9253	1	1	0.02	0.01400	0.700	1.00	1	3	0.0009	3.3009
13	5305 068 0505	8	8	0.87	0.00200	0.800	1.00	1	-1	0.8000	0.0
14	5305 068 0506	8	8	0.66	0.00200	0.800	1.00	1	-1	0.8000	0.0
15	3110 100 5348	1	1	4.81	0.00800	0.400	1.00	1	0	0.0703	0.6703
16	3110 144 8660	1	1	1.50	0.00800	0.400	1.00	1	0	0.0703	0.6703
17	3110 156 7721	1	1	0.39	0.00400	0.200	1.00	1	0	0.0187	0.8187
18	5330 171 9075	2	2	0.47	0.01400	1.400	1.00	1	2	0.0719	1.6719
19	3110 185 6530	1	1	1.01	0.01400	0.700	1.00	1	1	0.0408	1.3408
20	1015 219 8147	1	1	8.41	0.01400	0.700	1.00	1	0	0.1966	0.4966
21	1015 219 8166	1	1	1.54	0.05600	2.800	1.00	5	5	0.0169	[illegible].2169
22	1015 219 8177	1	1	0.86	0.01400	0.700	1.00	1	1	0.0408	1.3408
23	1015 219 8178	1	1	73.03	0.01600	0.800	1.00	1	-1	0.8000	0.0
24	5306 225 9088	14	14	0.02	0.00500	3.500	1.00	3	0	1.7277	0.2277
25	5305 269 2816	6	6	0.06	0.00500	1.500	1.00	1	0	0.7231	[illegible].2231

TABLE II - CONTINUED -

J	FSN			A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
26	5320	286	7399	2	2	0.20	0.01400	1.400	1.00	1	2	0.0719	1.6719
27	5330	298	5765	1	1	0.25	0.02800	1.400	1.00	1	4	0.0039	3.6039
28	1015	300	5386	1	1	23.33	0.00800	0.400	1.00	1	-1	0.4000	0.0
29	1015	300	5389	1	1	4.25	0.09600	4.800	1.00	5	6	0.0745	4.2745
30	1015	300	5390	1	1	5.87	0.05800	2.900	1.00	3	4	0.0580	3.1580
31	1015	300	5391	1	1	45.36	0.02800	1.400	1.00	1	0	0.6466	0.2466
32	1015	305	0754	1	1	88.98	0.00800	0.400	1.00	1	-1	0.4000	0.0
33	1005	321	8378	1	1	22.88	0.02400	1.200	1.00	1	0	0.5012	0.3012
34	5305	336	1616	1	1	0.29	0.01400	0.700	1.00	1	2	0.0066	2.3066
35	1015	340	3426	1	1	321.00	0.02800	1.400	1.00	1	0	0.6466	0.2466
36	1005	345	3826	1	1	7.75	0.01600	0.800	1.00	1	0	0.2493	0.4493
37	1005	345	3827	2	2	4.39	0.02000	2.000	1.00	2	1	0.3797	0.8797
38	1005	345	3828	1	1	16.69	0.02000	0.000	1.00	1	0	0.3679	0.3679
39	5330	350	9013	1	1	0.04	0.02800	1.400	1.00	1	5	0.0007	4.6007
40	5305	389	8133	6	6	0.29	0.00300	0.900	1.00	1	-1	0.9000	0.0
41	5310	407	9566	6	6	0.18	0.01000	3.000	1.00	3	1	0.7468	0.7468
42	1015	445	9902	1	1	2.09	0.01400	0.700	1.00	1	1	0.0408	1.3408
43	5315	474	7405	1	1	0.35	0.00800	0.400	1.00	1	1	0.0088	1.6088
44	1015	505	5249	2	2	18.09	0.00400	0.400	1.00	1	-1	0.4000	0.0
45	5315	505	5250	2	2	3.85	0.00400	0.400	1.00	1	-1	0.4000	0.0
46	1005	511	9052	1	1	1.47	0.04000	2.000	1.00	2	3	0.0488	2.5488
47	5315	543	3382	1	1	0.28	0.01400	0.700	1.00	1	2	0.0066	2.3066
48	5310	543	5752	2	2	0.10	0.01500	1.500	1.00	1	3	0.0242	2.5242
49	1015	587	2391	1	1	0.12	0.02600	1.300	1.00	1	4	0.0027	3.7027
50	1005	603	1297	1	1	1.15	0.01500	0.750	1.00	1	1	0.0490	1.2990
51	5315	616	5514	4	4	0.71	0.07700	15.400	1.00	15	15	0.2598	7.8598
52	5315	616	5522	2	2	1.38	0.00700	0.700	1.00	1	0	0.1966	0.4966
53	5305	638	7518	1	1	0.14	0.02800	1.400	1.00	1	4	0.0039	3.6039
54	5305	638	8869	6	6	0.03	0.01400	4.200	1.00	4	4	0.2210	2.5210
55	1015	672	8728	1	1	1.22	0.12000	6.000	1.00	6	9	0.0226	6.5226

TABLE II - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
56	1015 672 8729	1	1	0.80	0.01000	0.500	1.00	1	1	0.0163	1.5163
57	1015 672 8735	1	1	12.78	0.02400	1.200	1.00	1	1	0.1638	0.9638
58	1015 679 0481	1	1	6.47	0.02800	1.400	1.00	1	2	0.0719	1.6719
59	5305 716 8001	6	6	0.02	0.01000	3.000	1.00	3	2	0.3754	1.3754
60	1015 723 0796	1	1	2.13	0.02800	1.400	1.00	1	2	0.0719	1.6719
61	1015 723 0797	1	1	4.39	0.02000	0.000	1.00	1	1	0.1036	1.1036
62	1015 723 0799	1	1	5.35	0.02000	0.000	1.00	1	1	0.1036	1.1036
63	1017 723 0800	1	1	0.49	0.01400	0.700	1.00	1	1	0.0408	1.3408
64	5315 723 0802	2	2	0.29	0.12000	12.000	1.00	12	16	0.0265	10.5265
65	1015 723 0806	1	1	5.56	0.00800	0.400	1.00	1	0	0.0703	0.6703
66	1015 723 0807	2	2	0.31	0.02000	2.000	1.00	2	3	0.0488	2.5488
67	1015 723 0809	2	2	0.06	0.01900	1.900	1.00	2	4	0.0110	3.6110
68	1015 723 0810	1	1	0.12	0.12000	6.000	1.00	6	12	0.0015	9.5015
69	5305 724 5817	4	4	0.10	0.00800	1.600	1.00	2	1	0.2185	1.1185
70	5305 724 6799	2	2	0.02	0.01500	1.500	1.00	1	4	0.0056	3.5056
71	1005 726 6353	1	1	1.67	0.01400	0.700	1.00	1	1	0.0408	1.3408
72	1005 726 6357	1	1	7.36	0.02000	0.000	1.00	1	1	0.1036	1.1036
73	1005 726 6374	1	1	3.70	0.09600	4.800	1.00	5	6	0.0745	4.2745
74	5315 726 6380	3	3	0.63	0.02100	3.150	1.00	3	3	0.2012	2.0512
75	5315 726 6386	1	1	0.17	0.02400	1.200	1.00	1	3	0.0095	2.8095
76	1005 726 6400	1	1	3.18	0.01200	0.600	1.00	1	0	0.1488	0.5488
77	1005 726 6408	1	1	17.42	0.02000	0.000	1.00	1	0	0.3679	0.3679
78	1005 726 6413	1	1	0.06	0.01600	0.800	1.00	1	3	0.0016	3.2016
79	1005 726 6415	2	2	0.25	0.02000	2.000	1.00	2	3	0.0488	2.5488
80	1005 726 6416	1	1	0.09	0.01200	0.600	1.00	1	2	0.0038	2.4038
81	1005 726 6417	1	1	0.28	0.01400	0.700	1.00	1	2	0.0066	2.3066
82	1005 726 6419	1	1	0.13	0.02400	1.200	1.00	1	3	0.0095	2.8095
83	1005 726 6420	1	1	0.08	0.02400	1.200	1.00	1	4	0.0018	3.8018
84	1005 726 6421	2	2	0.12	0.01400	1.400	1.00	1	2	0.0719	1.6719
85	1005 726 6422	1	1	0.29	0.03400	1.700	1.00	2	4	0.0063	3.8063

TABLE II - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
86	5315 726 6497	1	1	0.52	0.01400	0.700	1.00	1	1	0.0408	1.3408
87	1005 726 6830	1	1	0.96	0.01200	0.600	1.00	1	1	0.0269	1.4269
88	5310 726 6831	1	1	3.98	0.01200	0.600	1.00	1	0	0.1488	0.5488
89	1015 730 7472	1	1	1.82	0.01400	0.700	1.00	1	1	0.0408	1.3408
90	1015 730 7475	1	1	0.16	0.00600	0.300	1.00	1	1	0.0039	1.7039
91	1015 730 7476	1	1	0.18	0.01400	0.700	1.00	1	2	0.0066	2.3066
92	1015 730 7477	1	1	1.23	0.01400	0.700	1.00	1	1	0.0408	1.3408
93	5305 730 7478	2	2	0.46	0.00200	0.200	1.00	1	-1	0.2000	0.0
94	1015 730 7490	1	1	4.36	0.01400	0.700	1.00	1	0	0.1966	0.4966
95	5310 730 7493	1	1	0.60	0.00800	0.400	1.00	1	0	0.0703	0.6703
96	1015 730 7494	1	1	3.25	0.00800	0.400	1.00	1	0	0.0703	0.6703
97	1015 730 7495	1	1	3.83	0.00800	0.400	1.00	1	0	0.0703	0.6703
98	1015 730 7496	1	1	3.51	0.00800	0.400	1.00	1	0	0.0703	0.6703
99	1015 730 7497	7	7	2.76	0.00400	1.400	1.00	1	-1	1.4000	0.0
100	5310 730 7498	2	2	0.26	0.00400	0.400	1.00	1	0	0.0703	0.6703
101	1015 730 7502	1	1	7.98	0.01400	0.700	1.00	1	0	0.1966	0.4966
102	1015 730 7503	1	1	9.33	0.01400	0.700	1.00	1	0	0.1966	0.4966
103	1015 730 7509	1	1	14.08	0.01400	0.700	1.00	1	0	0.1966	0.4966
104	1015 730 7511	1	1	5.12	0.01400	0.700	1.00	1	0	0.1966	0.4966
105	1015 730 7512	1	1	16.49	0.01400	0.700	1.00	1	0	0.1966	0.4966
106	1015 730 7513	1	1	6.72	0.01400	0.700	1.00	1	0	0.1966	0.4966
107	1005 730 7514	2	2	11.17	0.00700	0.700	1.00	1	-1	0.7000	0.0
108	1015 730 7517	1	1	37.69	0.00800	0.400	1.00	1	-1	0.4000	0.0
109	1015 730 7519	1	1	42.82	0.00800	0.400	1.00	1	-1	0.4000	0.0
110	1015 730 7523	1	1	26.69	0.01400	0.700	1.00	1	0	0.1966	0.4966
111	1005 751 9108	1	1	3.17	0.01200	0.600	1.00	1	0	0.1488	0.5488
112	1015 798 5164	1	1	6.09	0.01400	0.700	1.00	1	0	0.1966	0.4966
113	5340 826 1428	1	1	13.20	0.01200	0.600	1.00	1	0	0.1488	0.5488
114	5315 828 9829	2	2	0.45	0.01500	1.500	1.00	1	2	0.0898	1.5898
115	5340 837 6588	1	1	0.03	0.01400	0.700	1.00	1	3	0.0009	3.3009

TABLE II - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
116	5315 839 2325	6	6	0.11	0.00900	2.700	1.00	3	1	0.5830	0.8830
117	1015 840 3009	1	1	17.81	0.00800	0.400	1.00	1	-1	0.4000	0.0
118	5330 840 3010	1	1	0.49	0.05800	2.900	1.00	3	6	0.0065	5.1065
119	1015 840 3023	1	1	3.56	0.00800	0.400	1.00	1	0	0.0703	0.6703
120	5330 840 3025	1	1	0.11	0.05800	2.900	1.00	3	7	0.0019	6.1019
121	1015 840 3026	8	8	4.51	0.00700	2.800	1.00	3	-1	1.9176	0.1176
122	1015 840 3028	8	8	0.14	0.01700	6.800	1.00	7	4	0.8084	2.0084
123	1015 840 3029	1	1	3.80	0.00800	0.400	1.00	1	0	0.0703	0.6703
124	5330 840 3031	1	1	0.32	0.05800	2.900	1.00	3	6	0.0065	5.1065
125	1015 840 3056	1	1	7.97	0.02800	1.400	1.00	1	1	0.2384	0.8384
126	1015 840 3060	1	1	4.10	0.00800	0.400	1.00	1	0	0.0703	0.6703
127	1015 840 3062	1	1	2.62	0.02800	1.400	1.00	1	2	0.0719	1.6719
128	5315 840 3071	2	2	0.78	0.00700	0.700	1.00	1	0	0.1966	0.4966
129	5315 840 3072	2	2	0.48	0.00400	0.400	1.00	1	-1	0.4000	0.0
130	5315 840 3073	2	2	0.35	0.00400	0.400	1.00	1	-1	0.4000	0.0
131	1015 840 3086	4	4	0.96	0.01500	3.000	1.00	3	1	0.7468	0.7468
132	1015 840 3087	6	6	0.70	0.01000	3.000	1.00	3	0	1.3236	0.3236
133	1015 840 3185	1	1	3.10	0.01400	0.700	1.00	1	1	0.0408	1.3408
134	1015 840 3186	1	1	20.74	0.00800	0.400	1.00	1	-1	0.4000	0.0
135	1015 840 3187	1	1	0.74	0.02800	1.400	1.00	1	3	0.0182	2.6182
136	1015 840 3188	1	1	0.14	0.01400	0.700	1.00	1	2	0.0066	2.3066
137	5315 840 3189	1	1	0.52	0.01400	0.700	1.00	1	1	0.0408	1.3408
138	5330 840 3190	1	1	0.02	0.05800	2.900	1.00	3	8	0.0005	7.1005
139	5310 840 3192	1	1	0.25	0.00800	0.400	1.00	1	1	0.0088	1.6088
140	1015 840 3195	1	1	99.26	0.00800	0.400	1.00	1	-1	0.4000	0.0
141	1015 840 3242	1	1	2.59	0.01400	0.700	1.00	1	1	0.0408	1.3408
142	1015 840 3243	1	1	2.92	0.01400	0.700	1.00	1	1	0.0408	1.3408
143	5315 842 3308	3	3	1.26	0.06000	9.000	1.00	9	9	0.2060	5.2060
144	5315 844 3966	1	1	0.72	0.02800	1.400	1.00	1	3	0.0182	2.6182
145	1015 854 4471	2	2	0.02	0.03800	3.800	1.00	4	8	0.0031	6.7031

TABLE II - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
146	5315 898 9823	1	1	0.43	0.07600	3.800	1.00	4	7	0.0092	5.7092
147	1015 945 9769	1	1	0.30	0.06000	3.000	1.00	3	6	0.0080	5.0080
148	3110 948 9796	1	1	0.02	0.01400	0.700	1.00	1	3	0.0009	3.3009
149	5305 978 9378	3	3	0.03	0.02900	4.350	1.00	4	7	0.0220	5.1720
150	5305 983 6662	8	8	0.04	0.03700	14.800	1.00	15	14	0.2916	7.4916
151	5305 983 6664	1	1	0.04	0.02800	1.400	1.00	1	5	0.0007	4.6007
152	5305 983 6670	6	6	0.04	0.01000	3.000	1.00	3	2	0.3754	1.3754
153	5305 983 6671	6	6	0.04	0.01000	3.000	1.00	3	2	0.3754	1.3754
154	5305 983 6672	6	6	0.04	0.01000	3.000	1.00	3	2	0.3754	1.3754
155	5305 983 6673	2	2	0.07	0.01500	1.500	1.00	1	3	0.0242	2.5242
156	5305 983 7448	2	2	0.05	0.01400	1.400	1.00	1	3	0.0182	2.6182
157	5305 984 7363	8	8	0.35	0.00800	3.200	1.00	3	0	1.4815	0.2815
158	5315 988 8775	1	1	0.20	0.02800	1.400	1.00	1	4	0.0039	3.6039
159	5305 990 6444	3	3	0.30	0.02000	3.000	1.00	3	3	0.1682	2.1682

C. SAMPLE PROBLEM 2 - REQUIRED ASSURANCE AT LEAST K UNIT SYSTEMS UP CONSTRAINT

The second sample problem was computed for the required assurance at least k unit systems are up form of the constraint. Table III lists a summary of the input data and results for this problem.

TABLE III

SAMPLE PROBLEM 2 - SUMMARY OF INPUT DATA AND COMPUTED RESULTS

INPUT DATA	NUMBER OF UNIT SYSTEMS	50
	REQUIRED NUMBER OF UNIT SYSTEMS UP	47
	REQUIRED ASSURANCE	0.90
	OPERATING LEVEL	1.0
	ACTIVITY LEVEL	1.0
COMPUTED RESULTS	PROBABILITY AT LEAST K UNIT SYSTEMS ARE UP	0.91
	EXPECTED NUMBER OF UNIT SYSTEMS UP	47.46
	EXPECTED COST OF ON HAND INVENTORY	$387.88
	NUMBER OF ITERATIONS OF MARGINAL ANALYSIS PROCEDURE	348

The above data show that for the same system considered in Problem 1 with a required assurance of 0.90 that at least 47 unit systems are up program PKSMIN computed an r vector with an associated expected cost of on hand inventory of $387.88. The program overshot the required probability by 0.01 since the computed probability is 0.91 and the requirement was 0.90. After the r vector which met the constraint was computed the program computed the resulting value of $E(Z)$, the expected number of unit systems up. It is interesting to note that the expected number of unit systems up very nearly meets the requirement of Problem 1, and the expected cost of on hand inventory is also about the same as for Problem 1. The computer execution time for Problem 2 was about 19 seconds compared to slightly over four minutes for Problem 1.

Table IV lists the input paramaters and computed results for each FSN. The column headings are the same as for Problem 1.

D. EXPECTED COST OF ON HAND INVENTORY AS A FUNCTION OF REQUIRED AVAILABILITY

1. Expected Cost Versus Required Expected Number of Unit Systems Up

Program EZMIN was run for ten values of $A_s = 0.90, \ldots, 0.99$ in order to develop a functional relationship between the first form of the availability requirement and the expected cost of on hand inventory. Table V summaries the results. In addition to the expected cost of on hand inventory the table lists the number of iterations of the marginal analysis procedure required to find a solution and the number of reorder points which were less than the mean lead time demand for each level of A_s. A reorder point less than the mean lead time demand is equivalent to a negative safety level. For many of these the computed order quantity is one and the computed reorder point is minus one.

TABLE IV

SAMPLE PROBLEM 2 - REQUIRED ASSURANCE AT LEAST K UNIT SYSTEMS UP CONSTRAINT - INPUT PARAMETERS AND COMPUTED VALUES OF ORDER QUANTITIES, REORDER POINTS, EXPECTED BACKORDERS AND EXPECTED ON HAND QUANTITIES FOR ALL PART TYPES

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
1	5305 013 9135	1	1	0.05	0.01400	0.700	1.00	1	2	0.0066	2.3066
2	5305 013 9818	6	6	0.05	0.01500	4.500	1.00	4	-1	3.0823	0.0823
3	1005 047 3861	1	1	16.70	0.00800	0.400	1.00	1	-1	0.4000	0.0
4	1015 049 5241	2	2	0.42	0.00700	0.700	1.00	1	-1	0.7000	0.0
5	1015 049 5244	3	3	12.47	0.00300	0.450	1.00	1	-1	0.4500	0.0
6	5315 052 6697	1	1	0.65	0.02800	1.400	1.00	1	3	0.0182	2.6182
7	5315 058 6047	1	1	0.01	0.07200	3.600	1.00	4	10	0.0002	8.9002
8	5315 058 6065	1	1	0.01	0.07200	3.600	1.00	4	10	0.0002	8.9002
9	5315 058 6079	2	2	0.39	0.05600	5.600	1.00	5	7	0.0685	4.9685
10	5315 058 6093	1	1	0.01	0.01600	0.800	1.00	1	3	0.0016	3.2016
11	5315 058 6094	1	1	0.26	0.07200	3.600	1.00	4	[illegible]	0.0064	5.9064
12	5315 059 9253	1	1	0.02	0.01400	0.700	1.00	1	2	0.0066	2.3066
13	5305 068 0505	8	8	0.87	0.00200	0.800	1.00	1	-1	0.8000	0.0
14	5305 068 0506	8	8	0.66	0.00200	0.800	1.00	1	-1	0.8000	0.0
15	3110 100 5348	1	1	4.81	0.00800	0.400	1.00	1	-1	0.4000	0.0
16	3110 144 8660	1	1	1.50	0.00800	0.400	1.00	1	0	0.0703	0.6703
17	3110 156 7721	1	1	0.39	0.00400	0.200	1.00	1	-1	0.2000	0.0
18	5330 171 9075	2	2	0.47	0.01400	1.400	1.00	1	0	0.6466	0.2466
19	3110 185 6530	1	1	1.01	0.01400	0.700	1.00	1	1	0.0408	1.3408
20	1015 219 8147	1	1	0.41	0.01400	0.700	1.00	1	0	0.1966	0.4966
21	1015 219 8166	1	1	1.54	0.05600	2.800	1.00	3	5	0.0169	4.2169
22	1015 219 8177	1	1	0.86	0.01400	0.700	1.00	1	1	0.0408	1.3408
23	1015 219 8178	1	1	73.03	0.01600	0.800	1.00	1	-1	0.8000	0.0
24	5305 225 9088	14	14	0.02	0.00500	3.500	1.00	3	-1	2.5654	0.0654
25	5305 269 2816	6	6	0.06	0.00500	1.500	1.00	1	-1	1.5000	0.0

TABLE IV - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
26	5320 286 7399	2	2	0.20	0.01400	1.400	1.00	1	1	0.2384	0.8384
27	5330 298 5765	1	1	0.25	0.02800	1.400	1.00	1	3	0.0182	2.6182
28	1015 300 5386	1	1	23.33	0.00800	0.400	1.00	1	-1	0.4000	0.0
29	1015 300 5389	1	1	4.25	0.09600	4.800	1.00	5	7	0.0328	5.2328
30	1015 300 5390	1	1	5.87	0.05800	2.900	1.00	3	4	0.0580	3.1580
31	1015 300 5391	1	1	45.36	0.02800	1.400	1.00	1	1	0.2384	0.8384
32	1015 305 0754	1	1	88.98	0.00800	0.400	1.00	1	-1	0.4000	0.0
33	1005 321 8378	1	1	22.88	0.02400	1.200	1.00	1	0	0.5012	0.3012
34	5305 336 1616	1	1	0.29	0.01400	0.700	1.00	1	1	0.0408	1.3408
35	1015 340 3426	1	1	321.00	0.02800	1.400	1.00	1	0	0.6466	0.2466
36	1005 345 3826	1	1	7.75	0.01600	0.800	1.00	1	0	0.2493	0.4493
37	1005 345 3827	2	2	4.39	0.02000	2.000	1.00	2	0	0.8383	0.3383
38	1005 345 3828	1	1	16.69	0.02000	0.000	1.00	1	0	0.3679	0.3679
39	5330 350 9013	1	1	0.04	0.02800	1.400	1.00	1	4	0.0039	3.6039
40	5305 389 8133	6	6	0.29	0.00300	0.900	1.00	1	-1	0.9000	0.0
41	5310 407 9566	6	6	0.18	0.01000	3.000	1.00	3	-1	2.0996	0.0996
42	1015 445 9902	1	1	2.09	0.01400	0.700	1.00	1	0	0.1966	0.4966
43	5315 474 7405	1	1	0.35	0.00800	0.400	1.00	1	0	0.0703	0.6703
44	1015 505 5249	2	2	18.09	0.00400	0.400	1.00	1	-1	0.4000	0.0
45	5315 505 5250	2	2	3.85	0.00400	0.400	1.00	1	-1	0.4000	0.0
46	1005 511 9052	1	1	1.47	0.04000	2.000	1.00	2	3	0.0488	2.5488
47	5315 543 3382	1	1	0.28	0.01400	0.700	1.00	1	1	0.0408	1.3408
48	5310 543 5752	2	2	0.10	0.01500	1.500	1.00	1	1	0.2810	0.7810
49	1015 587 2391	1	1	0.12	0.02600	1.300	1.00	1	4	0.0027	3.7027
50	1005 603 1297	1	1	1.15	0.01500	0.750	1.00	1	1	0.0490	1.2990
51	5315 616 5514	4	4	0.71	0.07700	15.400	1.00	15	14	0.3770	6.9770
52	5315 616 5522	2	2	1.38	0.00700	0.700	1.00	1	-1	0.7000	0.0
53	5305 638 7518	1	1	0.14	0.02800	1.400	1.00	1	4	0.0039	3.6039
54	5305 638 8869	6	6	0.03	0.01400	4.200	1.00	4	-1	2.8028	0.1028
55	1015 672 8728	1	1	1.22	0.12000	6.000	1.00	6	10	0.0097	7.5097

TABLE IV - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
56	1015 672 8729	1	1	0.80	0.01000	0.500	1.00	1	0	0.1065	0.6065
57	1015 672 8735	1	1	12.78	0.02400	1.200	1.00	1	1	0.1638	0.9638
58	1015 679 0481	1	1	6.47	0.02800	1.400	1.00	1	2	0.0719	1.6719
59	5305 716 8001	6	6	0.02	0.01000	3.000	1.00	3	-1	2.0996	0.0996
60	1015 723 0796	1	1	2.13	0.02800	1.400	1.00	1	2	0.0719	1.6719
61	1015 723 0797	1	1	4.39	0.02000	0.000	1.00	1	1	0.1036	1.1036
62	1015 723 0799	1	1	5.35	0.02000	0.000	1.00	1	1	0.1036	1.1036
63	1017 723 0800	1	1	0.49	0.01400	0.700	1.00	1	1	0.0408	1.3408
64	5315 723 0802	2	2	0.29	0.12000	12.000	1.00	12	17	0.0144	11.5144
65	1015 723 0806	1	1	5.56	0.00800	0.400	1.00	1	-1	0.4000	0.0
66	1015 723 0807	2	2	0.31	0.02000	2.000	1.00	2	2	0.1466	1.6466
67	1015 723 0809	2	2	0.06	0.01900	1.900	1.00	2	2	0.1244	1.7244
68	1015 723 0810	1	1	0.12	0.12000	6.000	1.00	6	12	0.0015	9.5015
69	5305 724 5817	4	4	0.10	0.00800	1.600	1.00	2	-1	1.2009	0.1009
70	5305 724 6799	2	2	0.02	0.01500	1.500	1.00	1	2	0.0898	1.5898
71	1005 726 6353	1	1	1.67	0.01400	0.700	1.00	1	1	0.0408	1.3408
72	1005 726 6357	1	1	7.36	0.02000	0.000	1.00	1	1	0.1036	1.1036
73	1005 726 6374	1	1	3.70	0.09600	4.800	1.00	5	7	0.0328	5.2328
74	5315 726 6380	3	3	0.63	0.02100	3.150	1.00	3	1	0.8355	0.6855
75	5315 726 6386	1	1	0.17	0.02400	1.200	1.00	1	3	0.0095	2.8095
76	1005 726 6400	1	1	3.18	0.01200	0.600	1.00	1	0	0.1488	0.5488
77	1005 726 6408	1	1	17.42	0.02000	0.000	1.00	1	0	0.3679	0.3679
78	1005 726 6413	1	1	0.06	0.01600	0.800	1.00	1	2	0.0107	2.2107
79	1005 726 6415	2	2	0.25	0.02000	2.000	1.00	2	2	0.1466	1.6466
80	1005 726 6416	1	1	0.09	0.01200	0.600	1.00	1	1	0.0269	1.4269
81	1005 726 6417	1	1	0.28	0.01400	0.700	1.00	1	1	0.0408	1.3408
82	1005 726 6419	1	1	0.13	0.02400	1.200	1.00	1	3	0.0095	2.8095
83	1005 726 6420	1	1	0.08	0.02400	1.200	1.00	1	3	0.0095	2.8095
84	1005 726 6421	2	2	0.12	0.01400	1.400	1.00	1	1	0.2384	0.8384
85	1005 726 6422	1	1	0.29	0.03400	1.700	1.00	2	4	0.0063	3.8063

TABLE IV - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
86	5315 726 6497	1	1	0.52	0.01400	0.700	1.00	1	1	0.0408	1.3408
87	1005 726 6830	1	1	0.96	0.01200	0.600	1.00	1	1	0.0269	1.4269
88	5310 726 6831	1	1	3.98	0.01200	0.600	1.00	1	0	0.1488	0.5488
89	1015 730 7472	1	1	1.82	0.01400	0.700	1.00	1	1	0.0408	1.3408
90	1015 730 7475	1	1	0.16	0.00600	0.300	1.00	1	0	0.0408	0.7408
91	1015 730 7476	1	1	0.18	0.01400	0.700	1.00	1	2	0.0066	2.3066
92	1015 730 7477	1	1	1.23	0.01400	0.700	1.00	1	1	0.0408	1.3408
93	5305 730 7478	2	2	0.46	0.00200	0.200	1.00	1	-1	0.2000	0.0
94	1015 730 7490	1	1	4.36	0.01400	0.700	1.00	1	0	0.1966	0.4966
95	5310 730 7493	1	1	0.60	0.00800	0.400	1.00	1	0	0.0703	0.6703
96	1015 730 7494	1	1	3.25	0.00800	0.400	1.00	1	-1	0.4000	0.0
97	1015 730 7495	1	1	3.83	0.00800	0.400	1.00	1	-1	0.4000	0.0
98	1015 730 7496	1	1	3.51	0.00800	0.400	1.00	1	-1	0.4000	0.0
99	1015 730 7497	7	7	2.76	0.00400	1.400	1.00	1	-1	1.4000	0.0
100	5310 730 7498	2	2	0.26	0.00400	0.400	1.00	1	-1	0.4000	0.0
101	1015 730 7502	1	1	7.98	0.01400	0.700	1.00	1	0	0.1966	0.4966
102	1015 730 7503	1	1	9.33	0.01400	0.700	1.00	1	0	0.1966	0.4966
103	1015 730 7509	1	1	14.08	0.01400	0.700	1.00	1	0	0.1966	0.4966
104	1015 730 7511	1	1	5.12	0.01400	0.700	1.00	1	0	0.1966	0.4966
105	1015 730 7512	1	1	16.49	0.01400	0.700	1.00	1	0	0.1966	0.4966
106	1015 730 7513	1	1	6.72	0.01400	0.700	1.00	1	0	0.1966	0.4966
107	1005 730 7514	2	2	11.17	0.00700	0.700	1.00	1	-1	0.7000	0.0
108	1015 730 7517	1	1	37.69	0.00800	0.400	1.00	1	-1	0.4000	0.0
109	1015 730 7519	1	1	42.82	0.00800	0.400	1.00	1	-1	0.4000	0.0
110	1015 730 7523	1	1	26.69	0.01400	0.700	1.00	1	-1	0.7000	0.0
111	1005 751 9108	1	1	3.17	0.01200	0.600	1.00	1	0	0.1488	0.5488
112	1015 798 5154	1	1	6.09	0.01400	0.700	1.00	1	0	0.1966	0.4966
113	5340 825 1428	1	1	13.20	0.01200	0.600	1.00	1	-1	0.6000	0.0
114	5315 828 9829	2	2	0.45	0.01500	1.500	1.00	1	1	0.2810	0.7810
115	5340 837 6588	1	1	0.03	0.01400	0.700	1.00	1	2	0.0066	2.3066

TABLE IV - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
116	5315 839 2325	6	6	0.11	0.00900	2.700	1.00	3	-1	1.8277	0.1277
117	1015 840 3009	1	1	17.81	0.00800	0.400	1.00	1	-1	0.4000	0.0
118	5330 840 3010	1	1	0.49	0.05800	2.900	1.00	3	6	0.0065	5.1065
119	1015 840 3023	1	1	3.56	0.00800	0.400	1.00	1	-1	0.4000	0.0
120	5330 840 3025	1	1	0.11	0.05800	2.900	1.00	3	7	0.0019	6.1019
121	1015 840 3026	8	8	4.51	0.00700	2.800	1.00	3	-1	1.9176	0.1176
122	1015 840 3028	8	8	0.14	0.01700	6.800	1.00	7	-1	3.9682	0.1682
123	1015 840 3029	1	1	3.80	0.00800	0.400	1.00	1	-1	0.4000	0.0
124	5330 840 3031	1	1	0.32	0.05800	2.900	1.00	3	6	0.0065	5.1065
125	1015 840 3056	1	1	7.97	0.02800	1.400	1.00	1	1	0.2384	0.8384
126	1015 840 3060	1	1	4.10	0.00800	0.400	1.00	1	-1	0.4000	0.0
127	1015 840 3062	1	1	2.62	0.02800	1.400	1.00	1	2	0.0719	1.6719
128	5315 840 3071	2	2	0.78	0.00700	0.700	1.00	1	-1	0.7000	0.0
129	5315 840 3072	2	2	0.48	0.00400	0.400	1.00	1	-1	0.4000	0.0
130	5315 840 3073	2	2	0.35	0.00400	0.400	1.00	1	-1	0.4000	0.0
131	1015 840 3086	4	4	0.96	0.01500	3.000	1.00	3	-1	2.0996	0.0996
132	1015 840 3087	6	6	0.70	0.01000	3.000	1.00	3	-1	2.0996	0.0996
133	1015 840 3185	1	1	3.10	0.01400	0.700	1.00	1	0	0.1966	0.4966
134	1015 840 3186	1	1	20.74	0.00800	0.400	1.00	1	-1	0.4000	0.0
135	1015 840 3187	1	1	0.74	0.02800	1.400	1.00	1	3	0.0182	2.6182
136	1015 840 3188	1	1	0.14	0.01400	0.700	1.00	1	2	0.0066	2.3066
137	5315 840 3189	1	1	0.52	0.01400	0.700	1.00	1	1	0.0408	1.3408
138	5330 840 3190	1	1	0.02	0.05800	2.900	1.00	3	8	0.0005	7.1005
139	5310 840 3192	1	1	0.25	0.00800	0.400	1.00	1	0	0.0703	0.6703
140	1015 840 3195	1	1	99.26	0.00800	0.400	1.00	1	-1	0.4000	0.0
141	1015 840 3242	1	1	2.59	0.01400	0.700	1.00	1	0	0.1966	0.4966
142	1015 840 3243	1	1	2.92	0.01400	0.700	1.00	1	0	0.1966	0.4966
143	5315 842 3308	3	3	1.26	0.06000	9.000	1.00	9	8	0.3373	4.3373
144	5315 844 3966	1	1	0.72	0.02800	1.400	1.00	1	3	0.0182	2.6182
145	1015 854 4471	2	2	0.02	0.03800	3.800	1.00	4	7	0.0092	5.7092

TABLE IV - CONTINUED -

J	FSN	A	B	C	RHOHAT	RHO	TAU	Q	R	EBO	EOH
146	5315 898 9823	1	1	0.43	0.07600	3.800	1.00	4	7	0.0092	5.7092
147	1015 945 9769	1	1	0.50	0.06000	3.000	1.00	3	6	0.0080	5.0080
148	3110 948 9796	1	1	0.02	0.01400	0.700	1.00	1	2	0.0066	2.3066
149	5305 978 9378	3	3	0.03	0.02900	4.350	1.00	4	5	0.1220	3.2720
150	5305 983 6662	8	8	0.04	0.03700	14.800	1.00	15	6	2.7712	1.9712
151	5305 983 6664	1	1	0.04	0.02800	1.400	1.00	1	4	0.0039	3.6039
152	5305 983 6670	6	6	0.04	0.01000	3.000	1.00	3	-1	2.0996	0.0996
153	5305 983 6671	6	6	0.04	0.01000	3.000	1.00	3	-1	2.0996	0.0996
154	5305 983 6672	6	6	0.04	0.01000	3.000	1.00	3	-1	2.0996	0.0996
155	5305 983 6673	2	2	0.07	0.01500	1.500	1.00	1	1	0.2810	0.7810
156	5305 983 7448	2	2	0.05	0.01400	1.400	1.00	1	1	0.2384	0.8384
157	5305 984 7363	8	8	0.35	0.00800	3.200	1.00	3	-1	2.2842	0.0842
158	5315 988 8775	1	1	0.20	0.02800	1.400	1.00	1	4	0.0039	3.6039
159	5305 990 6444	3	3	0.30	0.02000	3.000	1.00	3	1	0.7468	0.7468

This (Q,r) implies that the part is not stocked by the maintenance unit but is ordered from the source of supply as demands occur.

The data show that expected costs of on hand inventory rise very steeply as the required level of A_s approaches unity.

TABLE V

EXPECTED COST OF ON HAND INVENTORY VERSUS REQUIRED A_s

A_s	E(C)	No. of Iterations	No. of Negative Safety Levels
0.90	$16.80	31	156
0.91	$22.02	64	151
0.92	$38.14	67	143
0.93	$79.57	140	123
0.94	$168.64	141	104
0.95	$418.04	295	73
0.96	$645.76	238	41
0.97	$1225.36	359	19
0.98	$1619.00	322	7
0.99	$2406.83	386	2

Figure 1 plots the expected cost of on hand inventory E(C) as a function of A_s.

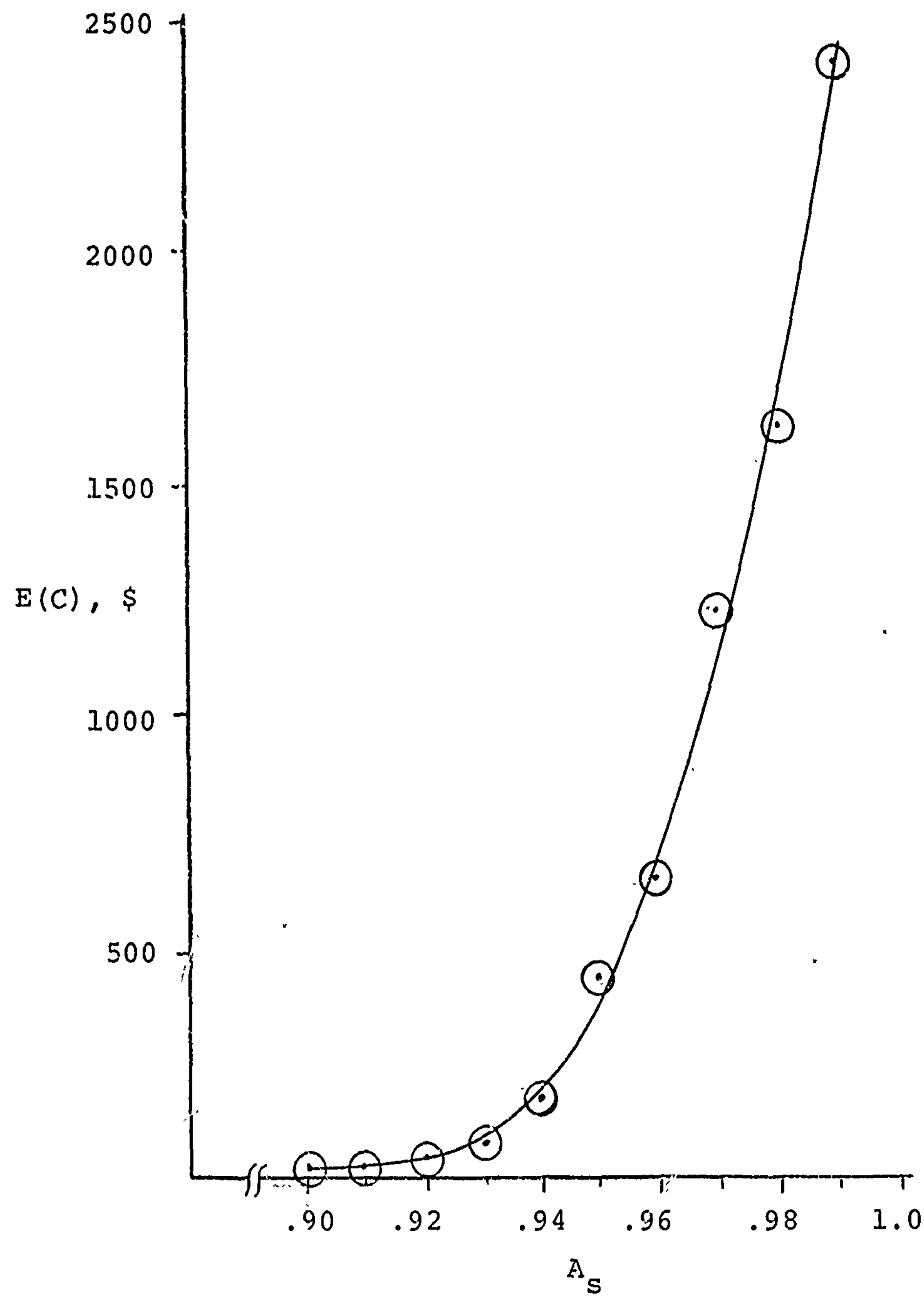


Figure 1. Expected Cost of on Hand Inventory E(C) as a Function of Required Expected Fraction of Unit Systems Up

2. <u>Expected Cost of On Hand Inventory Versus Required Number of Unit Systems Up with an Assurance of 90 Per Cent</u>

Program PKSMIN was run for six values of k = 45, ...,50 at a required assurance of 90 per cent in order to develop a functional relationship between the required value of k and the expected cost of on hand inventory at this assurance level. Table VI summaries the results.

TABLE VI

EXPECTED COST OF ON HAND INVENTORY VERSUS REQUIRED NUMBER UNIT SYSTEMS UP WITH 90 PER CENT ASSURANCE

k	E(C)	No. of Iterations	No. of Negative Safety Levels
45	$29.07	103	150
46	$84.97	102	122
47	$387.88	177	85
48	$1117.47	218	49
49	$2235.32	278	11
50	$3519.92	303	0

Figure 2 plots the expected cost of on hand inventory E(C) as a function of the required number k of unit systems up with 90 per cent assurance.

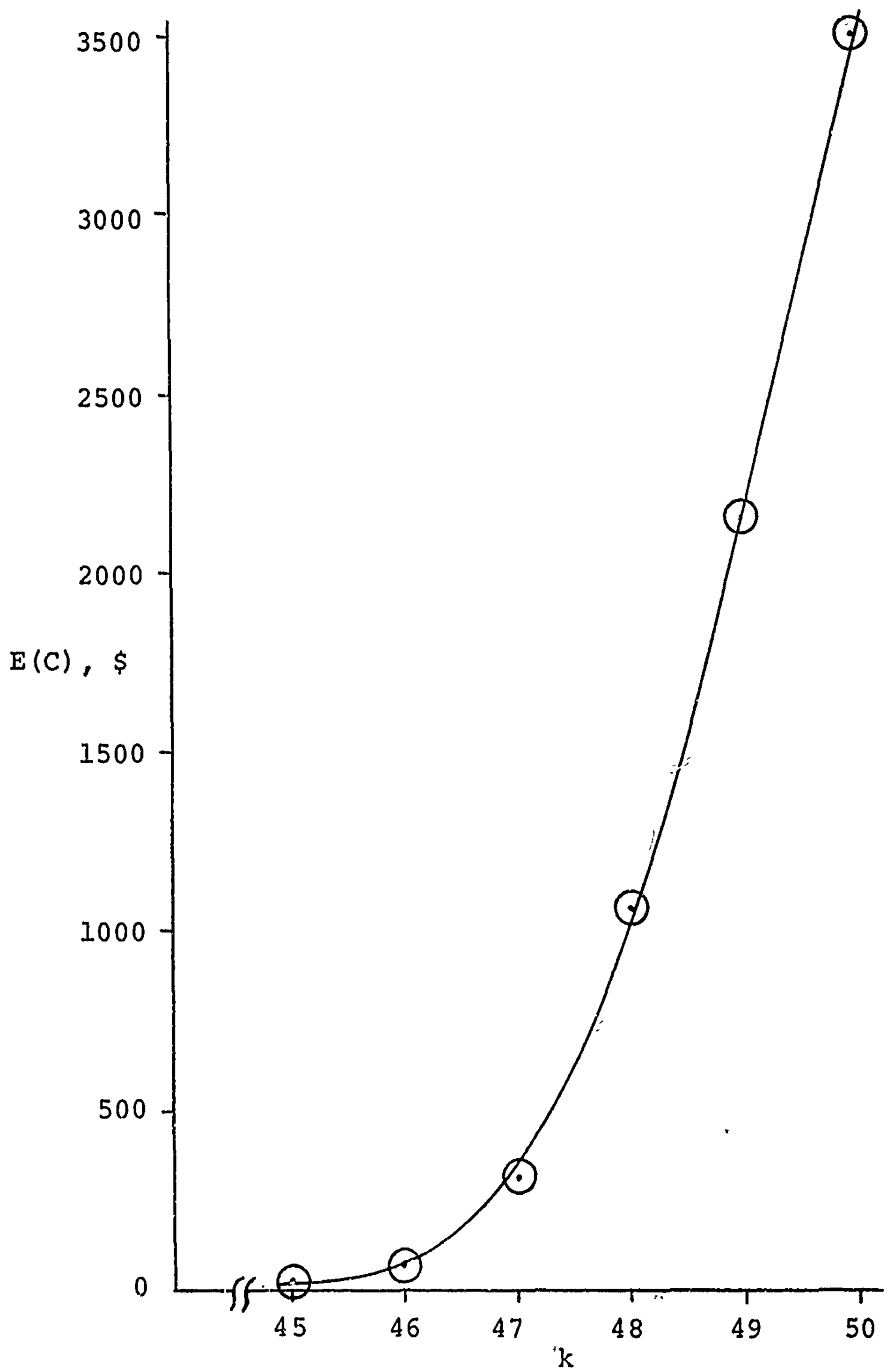


Figure 2. Expected Cost of on Hand Inventory E(C) as a Function of the Required Number k of Unit Systems Up with 90 Per Cent Assurance

V. OBSERVATIONS AND CONCLUSIONS

A. OBSERVATIONS

1. Marginal Analysis may Produce only a Near-Optimal Solution

The marginal analysis procedure presented in Section III sometimes overshoots the optimum solution. That is, the $\underline{r}$ vector found using the procedure sometimes results in a higher than required availability and a higher expected cost of on hand inventory than needed to satisfy the constraint. This behavior of marginal analysis was observed during the hand calculation of the solution to a small problem with four unit systems and four part types where the constraint was $E(Z) \geq 3.6$. Marginal analysis yielded an $E(Z)$ of 3.709 and an $E(C)$ of \$53.87. It was possible, however, on the last iteration to achieve an $E(Z)$ of 3.601 at a cost of \$48.63 by increasing a different r_j than the one associated with the maximum Δ_j. A dynamic programming procedure would overcome this shortcoming, but dynamic programming rapidly becomes computationally cumbersome as the number of decision variables increases. In the sample problems discussed in Section IV there were 159 decision variables, and in many possible applications of the model the number would be in the thousands. Thus the simple marginal analysis procedure may be more computationally feasible than dynamic programming for many applications of interest. Further, the relative differences between the optimal solution and the marginal analysis solution tends to decrease as the size of the problem increases.

2. <u>Program PKSMIN is Much Faster than Program EZMIN</u>

Sample problems 1 and 2 in Section IV resulted in about the same values of $E(Z)$ and of $E(C)$. For problem 1 $E(Z)$ was 47.58 and $E(C)$ was \$418.04. For problem 2 $E(Z)$ was 47.46 and $E(C)$ was \$387.88. On the IBM 360 program EZMIN required over four minutes to compute the results for problem 1, while program PKSMIN solved problem 2 in nineteen seconds. The reason for this difference is that $E(Z)$ a more complex function than $Pr(Z \geq k)$. For a given $\underline{r}$ vector $E(Z)$ is the sum over k from one to S of $Pr(Z \geq k)$. A comparison of the JMAX segments of subroutines OPTIMR and OPTMZ shows that OPTIMR requires more computations at each iteration (See Flow charts B4a and C3a.)

3. <u>Functional Relationships of Expected Cost Versus Required Availability are Readily Obtained</u>

The functional relationships of expected cost of on hand inventory versus availability discussed in Section IV were obtained by simply adding a DO loop to the two main programs. Thus for $E(C)$ versus A_s the value of A_s was initially set to 0.89, and at each iteration of the DO loop A_s was increased by 0.01 for ten iterations. Similarly for $E(C)$ versus k at the 90 per cent assurance level k was initially set at 44 and incremented by one at each iterations for six iterations.

Using the functional relationships of $E(C)$ versus availability we can estimate the maximum availability which can be achieved if there is a constraint on $E(C)$. Suppose, for example, we desired to maximize A_s subject to $E(C) \leq \$1000.00$. From Table V in Section IV we see that the maximum A_s would be between .96 and .97. To determine maximum A_s more precisely and to determine the associated $\underline{r}$ vector we could run program EZMIN for a series of values of A_s, say .961, .962, ..., .969. The resulting $E(C)$ for one of these A_s values should be very close to \$1000.00.

Computing time in developing the $E(C)$ versus availability functions can be conserved if at each iteration after the first the starting values of the r_j are the final values computed on the previous iteration. This can be accomplished if the initializing subroutines INITAL and INTLZ are skipped on the second and subsequent iterations. This technique reduces the computing time to about half that needed if the initializing subroutines are called at each iteration.

B. CONCLUSIONS

1. Advantages of the Model

The model provides the first technique of which the author is aware for explicitly taking into account system availability in the determination of continuous review inventory policies for repair parts. The model allows for variable safety levels and yields a set policies where the ratio of marginal availability to marginal cost of inventory investment is approximately the same for all part types. The result is that the model tends to favor low-cost, high-demand items with relatively high safety levels. High-cost, low-demand items are less well protected. All items are, however, well enough protected to insure, under the assumptions of the model, that the availability requirement will be met.

Compared to models which might more realistically represent the "real world" the one presented here possesses the advantage of computational feasibility. Programs EXMIN and PKSMIN produce solutions in a reasonable amount of time. Refinements in the programming could probably improve run times.

2. Limitations of the Model

The limitations stem primarily from the assumptions which were made for the sake of mathematical simplicity. For example, the constant lead time assumption is not usually true. Lead times can have considerable variability. Lead time variability will tend to decrease the actual availabilities achieved compared to those predicted by the model. On the other hand the model does not consider the fact that requisitions for parts which are causing a unit system to be down have a higher priority and thus a shorter lead time than normal replenishment requisitions. This fact will tend to make achieved availabilities higher than those predicted.

The optimal cannibalization assumption is seldom completely true. The significance of this assumption decreases, however, as the required availability increases, since at higher availability levels we expect fewer backorders and therefore fewer occasions for cannibalization.

The significance of the assumption of zero or one states for unit systems will depend upon the stringency of the serviceability criteria used to determine whether a unit system is up or down. The more stringent the criteria, the more parts will have b_j equal to a_j. Parts with b_j equal to a_j will generally be better protected than parts with b_j less than a_j. The most stringent criteria possible would be that every part on the unit system must be working if it is to be counted as being up. Thus we can see that the inventory policies computed by this model and the resulting expected costs of on hand inventory are highly dependent upon the serviceability criteria for the unit system.

The assumption that parts are replaced immediately with zero lead time makes it necessary to correct the availabilities computed by the model for down time due to finite time to repair.

3. Uses of the Model

The model was developed for use in determining direct support inventory policies for repaire parts for a given population of a particular major item.

Another use which suggests itself is determining optimal maintenance floats. Suppose it were required that at least k unit systems be up with probability .90. Assume that the capability of the maintenance unit to make repairs, given that needed parts are available, is fixed. Then the probability that k unit systems are up is determined by the total number S of unit systems and the inventory policies for repair parts. The quantity S - k is called the maintenance float. An interesting problem is determining the optimal level of the maintenance float. The model presented in this thesis could be used to estimate the optimal float level as follows. Compute the expected cost of on hand inventory needed to assure with probability .90 that k unit systems are up with zero, one, two, ... unit systems in the maintenance float. Then choose that level of maintenance float which minimizes the sum of the cost of the float plus the on hand inventory.

4. Extensions

A number of possible extensions of the model appear to be worthy of investigation. For example, relaxation of the assumption that all unit systems are identical would be useful. A next higher level of complexity of the system would be one in which there are two types of unit systems. For example, one type of unit system might be a particular type of weapon. The other unit system might be a piece of fire control equipment which controls the fires of several weapons. If a fire control unit is down all the weapons it controls are also down. Application of the approach of this model to optimal inventory policies for such a system seems to offer interesting possibilities.

In any case it is hoped that the model presented here will prove useful in the development of system availability-oriented operating policies for military repair parts inventories.

APPENDIX A

CONVENXITY CONSIDERATIONS

In order to insure the optimality of the $\underline{r}$ vector found using the procedure outlined in Section III it is necessary that the cost function be convex and that the region in $\underline{r}$-space defined by the constraint be convex.

1. The Cost Function

The objective function in the model presented in this paper is the expected cost of on hand inventory, which is given by

$$D(C) = \sum_{j=1}^{N} C_j D\ (Q_j, r_j).$$

Since the Q_j are fixed by the operating level of supply policy as described in Section III, we are concerned only with convexity with respect to the r_j, $j = 1, \ldots, N$. Further, since $E(C)$ is a sum of terms each of which is a function only of r_j, $E(C)$ is convex if each $D_j(Q_j, r_j)$ is convex. Since Q_j is fixed let us drop it as an argument; also let us fix j and drop the subscript. Hadley and Whitin [Ref. 8, p. 184] show that

$$D(r) = (Q + 1)/2 + r + B(Q,r),$$

where $B(Q,r)$ is the expected backorders, given a (Q,r) policy. $D(r)$ is linear in r except for the $B(Q,r)$ term.

Thus $D(r)$ is convex if $B(Q,r)$ is convex in r. Again, since Q is fixed, let us drop it as an argument. Hadley and Whitin [Ref. 8, p. 184] show that

$$B(r) = (1/Q) \sum_{y=0}^{\infty} y[P(y+r+1;\mu) - P(y+r+Q+1;\mu)].$$

Let

$$\Delta_r B(r) = B(r+1) - B(r),$$

and

$$\Delta_r^2 B(r) = \Delta_r B(r+1) - \Delta_r B(r).$$

$D(r)$ is convex if $\Delta_r^2 B(r) \leq 0$ for all r. Since the parameter μ is fixed for fixed j, let us further simplify the notation by letting

$$P(x) = P(x;\mu),$$

$$p(x) = p(x;\mu).$$

Further, let

$$B^*(r) = QB(r).$$

Then $D(r)$ is convex for $r \geq r^*$ if $\Delta_r^2 B^*(r) \geq 0$ for all $r \geq r^*$. Now

$$\Delta_r B^*(r) = \sum_{y=0}^{\infty} y \begin{bmatrix} P(y+r+2) - P(y+r+Q+2) \\ -P(y+r+1) + P(y+r+Q+1) \end{bmatrix}$$

$$= \sum_{y=1}^{\infty} y[p(y+r+Q+1) - p(y+r+1)].$$

Note that we nave dropped the term for $y = 0$ since it contributes nothing to the sum.

Now

$$\Delta_r^2 B^*(r) = \sum_{y=1}^{\infty} y \begin{bmatrix} p(y+r+Q+2) & - p(y+r+2) \\ -p(y+r+Q+1) & + P(y+r+1) \end{bmatrix}$$

$$= \sum_{y=1}^{\infty} y[\Delta p(y+r+Q+1) - \Delta p(y+r+1)]$$

where

$$\Delta p(x) = p(x+1) - p(x)$$

$$= \frac{\mu^{(x+1)} e^{-\mu}}{(x+1)!} - \frac{\mu^x e^{-\mu}}{x!}$$

$$= [\frac{\mu - x - 1}{x + 1}] \; p(x).$$

Thus

$$\Delta_r^2 B^*(r) = \sum_{y=1}^{\infty} y[\frac{\mu-y-r-Q-2}{y+r+Q+2} p(y+r+Q+1) - \frac{\mu-y-r-2}{y+r+2} p(y+r+1)]$$

$$= \sum_{y=1}^{\infty} y[\frac{y+r+2-\mu}{y+r+2} p(y+r+1) - \frac{y+r+Q+2-\mu}{y+r+Q+2} p(y+r+Q+2)$$

$$\doteq \begin{bmatrix} \frac{(r+3-\mu)}{r+3} p(r+2) & - \frac{(r+Q+3-\mu)}{r+Q+3} p(r+Q+2) \\ + \frac{2(r+4-\mu)}{r+4} p(r+3) & - \frac{2(r+Q+4-\mu)}{r+Q+4} p(r+Q+3) \\ \vdots & \\ + (Q+1) \frac{(r+Q+3)}{r+Q+3} p(r+Q+2) & - \ldots \\ + \ldots & \end{bmatrix}$$

$$= \sum_{y=1}^{Q-1} y\left(\frac{r+y+2-\mu}{r+y+2}\right)p(y+r+1) + Q\sum_{y=Q}^{\infty}\left(\frac{r+y+2-\mu}{r+y+2}\right)p(r+y+1).$$

Now every term in this series is non-negative if $r \geq \mu-3$. Thus $D(r)$ is convex in r for $r = \mu-3$. Furthermore, numerical calculations show that for values of μ up to 40 that if $Q \geq \mu-.5$ then $D(r)$ is convex for $r \geq -1$. This numerical result is probably due to the fact that the mode of the Poisson probability mass function is approximately $\mu-1$, which means that the most negative values of $(r+y+2-\mu)/(r+y+2)$ in the series are multiplied by small values of $p(x;\mu)$. When $r \geq -1$, all terms in the series for $y \geq \mu - 1$ will be positive. Thus it is likely that the sum of the positive terms in the series is greater than the absolute value of the sum of the terms for which $y + 1 - \mu$ is negative.

Thus, as far as the cost function is concerned, we can be sure that the function is convex in $\underline{r}$ for all $\underline{r} \geq \underline{\mu} - \underline{3}$, where $\underline{\mu} = \mu_1,\ldots,\mu_N$, and we have reason to believe the function is convex for $\underline{r} \geq -\underline{1}$ when $Q_j \geq \mu_j - .5$, $j = 1,\ldots,N$.

2. The Constraint Function

The two alternate forms of the constraint functions are, first, a required expected number of operational unit systems, expressed by

$$\sum_{k=1}^{S}\prod_{j=1}^{N}\sum_{y_j=0}^{(Sa_j-kb_j)}\Psi_j(y_j;r) \geq A_s S\ ,$$

and second a required probability that at least k unit systems are operational, expressed by

$$\prod_{j=1}^{N} \sum_{y_j=0}^{Sa_j-kb_j} \Psi_j(y;r_j) = P_{min}(k) \quad .$$

If the constraint region is to be convex, the above functions must be concave functions of $\underline{r} = (r_1,\ldots,r_N)$. Note that the left hand side of the first constraint is the sum over k of terms of the same form as the left hand side of the second constraint. Thus if we can show the conditions for concavity of the left hand side of the second constraint for a general k, we can easily show the conditions for concavity of the first constraint.

The difficulty in showing the concavity of the second form of constraint is that it is not a separable function of the r_j. It can be transformed into a separable function by taking the logarithm, in which case it becomes a sum of terms, each of which is a function of only one r_j, i.e.,

$$\ln[\Pr\{Z \geq k\}] = \sum_{j=1}^{N} \ln[\Pr\{Z_j \geq k\}],$$

where

$$\Pr\{Z_j \geq k\} = \sum_{y_j=0}^{Sa_j-kb_j} \Psi_j(y_j;r_j).$$

Let

$$f(\underline{r}) = \Pr\{Z \geq k;\underline{r}\}, = \prod_{j=1}^{N} f_j(r_i),$$

$$f_j(r_j) = \Pr\{Z_j \geq k;r_j\},$$

$$\phi(\underline{r}) = \ln f(\underline{r}),$$

$$\phi_j(r_j) = \ln f_j(r_j).$$

Now it can easily be shown that if $f_j(r_j)$ is concave, $\phi_j(r_j)$ is also concave, and thus $\phi(\underline{r})$ is concave. Unfortunately, the concavity of $\phi(\underline{r})$ does not imply the concavity of $f(\underline{r})$. Now consider

$$f_j(r_j) = \sum_{y_j=0}^{Sa_j-kb_j} \Psi_j(y_j;r_j).$$

Let

$$M = Sa_j - kb_j,$$

then

$$f_j(r_j) = \Psi_j(0;r_j) + \ldots + \Psi_j(M;r_j).$$

In section III we showed that

$$\Psi_j(0;r_j) = \sum_{y_j=0}^{r_j+1} \Psi_j(y_j;-1),$$

$$\Psi_j(y_j;r_j) = \Psi_j(y_j + r_j + 1; - 1), \; y_j > 0.$$

Thus we have

$$f_j(r_j) = \sum_{u=0}^{r_j+1} \Psi_j(u;-1) + \sum_{y=1}^{M} \Psi_j(y_j+r_j+1;-1)$$

$$= \sum_{u=0}^{M+r_j+1} \Psi_j(u;-1).$$

Now let

$$\Delta_{r_j} f_j(r_j) = f_j(r_j+1) - f_j(r_j),$$

then

$$\Delta_{r_j} f(r_j) = \sum_{u=0}^{M+r_j+2} \Psi_j(u;-1) - \sum_{u=0}^{M+r_j+1} \Psi_j(u;-1)$$

$$= \Psi_j(M+r_j+2;-1).$$

Let

$$\Delta^2_{r_j} f_j = \Delta_{r_j} f(r_j+1) - \Delta_{r_j} f(r_j).$$

then

$$\Delta^2_{r_j} f_j = \Psi_j(M+r+3;-1) - \Psi_j(M+r+2;-1)$$

$$= (1/Q)\begin{bmatrix} P(M+r_j+3) & - P(M+Q_j+r_j+3) \\ -P(M+r+2) & - P(M+Q+r+2) \end{bmatrix}$$

$$= (1/Q)\,[p(M+Q+r+2) - p(M+r+2)]$$

Now f_j is concave for all $r_j \geq r_j^*$ if the expression in brackets above is ≤ 0 for all $r_j \geq r_j^*$. Thus we seek r_j^* such that for all $r_j \geq r_j^*$

$$p(M+r_j+2) \geq p(M+Q_j+r_j+2)$$

or

$$\frac{\mu_j^{(M+r_j+2)} e^{-\mu}}{(M+r_j+2)!} \geq \frac{\mu_j^{(M+Q_j+r_j+2)} e^{-\mu}}{(M+Q_j+r_j+2)!}$$

or

$$\mu^{Q_j} \leq \frac{(M+r_j+Q_j+2)!}{(M+r_j+2)!} \, .$$

The right hand side of the above inequality is an increasing function of r_j, Q_j, and M. It is usually true that the operating level policy used within the Army is at least as many days of supply as the procurement lead time. This means that $Q_j \geq u_j$. Assume this is true. Now if $k = S$ and $a_j = b_j$, then $M = 0$. If $f_j(r_j;k=S)$ is concave, $f_j(r_j;k<S)$ will also be concave, since if the inequality holds for $M = 0$, it will hold for $M>0$. Let $M = 0$. Then the inequality becomes

$$\mu^{Q_j} \leq \frac{(r_j+Q_j+2)!}{(r_j+2)!}$$

For example, if $r_j = -1$ and $Q_j = 1$, then μ_j must be equal to or less than 2. In other words, for $u_j \leq 2$, $f_j(r_j)$ is concave in r_j for all $r_j \geq -1$, $Q_j \geq 1$, and $k \leq S$.

To insure the concavity of $f_j(r_j)$ for $\mu > 2$, in the FORTRAN program for the $E(Z) = A_S S$ form of the constraint, the subroutine INITAL finds an initial set of r_j such that the concavity conditions for $k = S$ are satisfied for r_j equal to or greater than the initial r_j. Similarly, in the FORTRAN program for the $Pr(Z=k) = P_{min}$ form of the constraint, subroutine INTLZ finds an initial set of r_j such that each $Pr\{Z_j = k; r_j)$ is a concave function of r_j for r_j equal to or greater than the initial value.

The solution procedure employed in the computer programs thus insures that each $f_j(r_j)$ is concave in r_j in the region that the marginal analysis procedure searches for an optimum. The question of the concavity of $f(\underline{r}) = \prod_{j=1}^{N} f_j(r_j)$ remains open. To date the numerical results have not indicated that there is a problem, i.e., the numerical results are quite reasonable. No proof, however, of the concavity of $f(\underline{r})$ has been found.

APPENDIX B

FLOW CHARTS FOR FORTRAN PROGRAM EZMIN

FORTRAN program EZMIN was developed to solve the problem of finding the vector $\underline{r}$ which minimizes the expected cost of on hand inventory subject to a required expected number of unit systems up. This appendix contains flow charts for the main program and the following subprograms:

Subroutine PSITAB, which computes a table of values of PI(J,Y) for $j = 1,\ldots,N$, and $Y = 1,\ldots,100$, where

$$PI(J,Y) = \Psi_J(Y-1; r_J = -1) .$$

Subroutine INITAL, which computes a set of initial values of r_j, such that $E(Z_j) \geq E(Z)_{min}$, $j = 1,\ldots,N$.

Subroutine OPTIMR, which uses a marginal analysis technique to find an optimal $\underline{r}$ vector.

Function PRZJK, which computes $Pr\{Z_j = k; r_j\}$.

Function EBO, which computes $E(Y_j; r_j)$.

Function PPT, which computes individual terms of the Poisson probability distribution.

Function PCUMT, which computes complementary cumulative terms of the Poisson probability distribution.

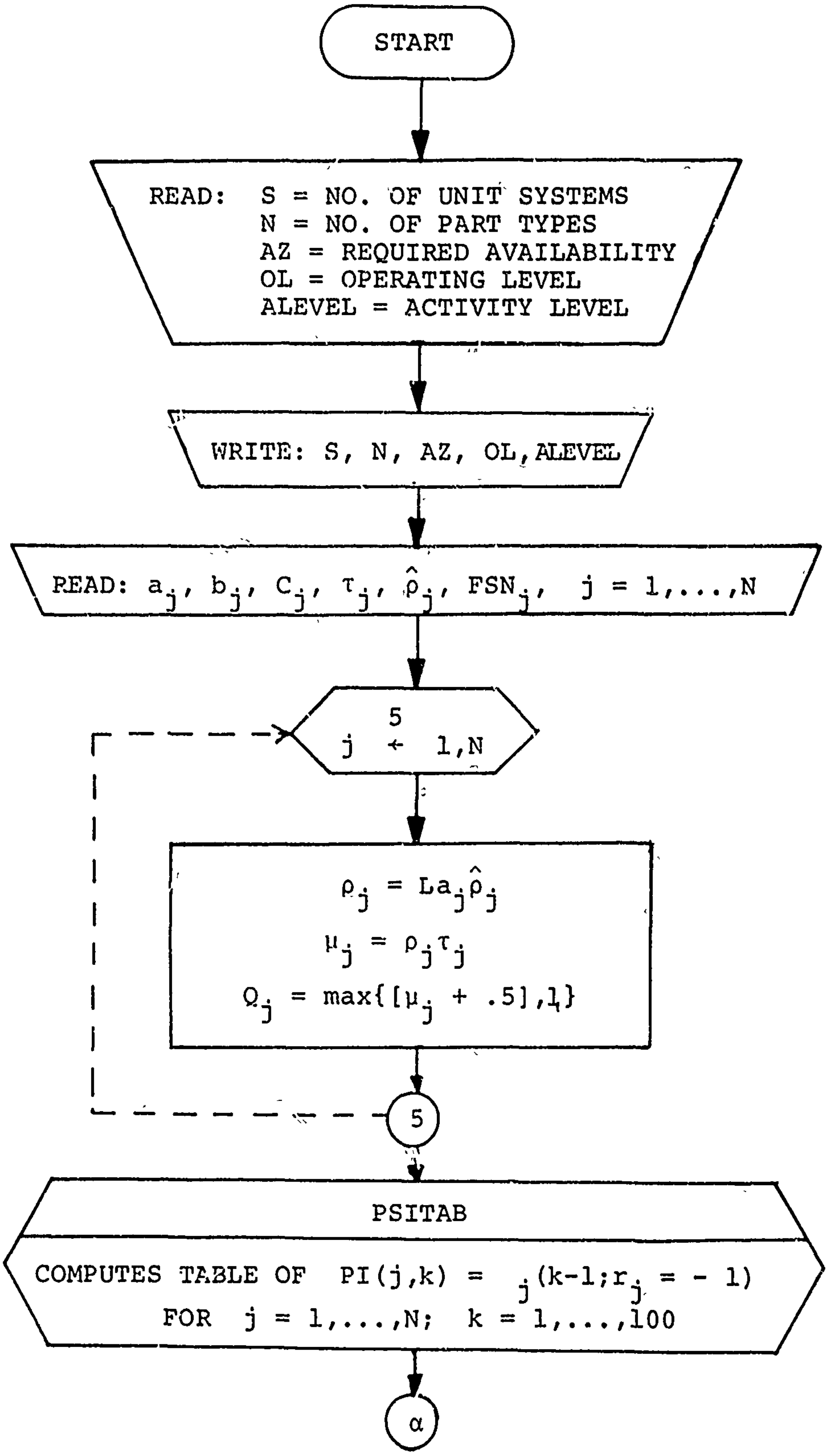
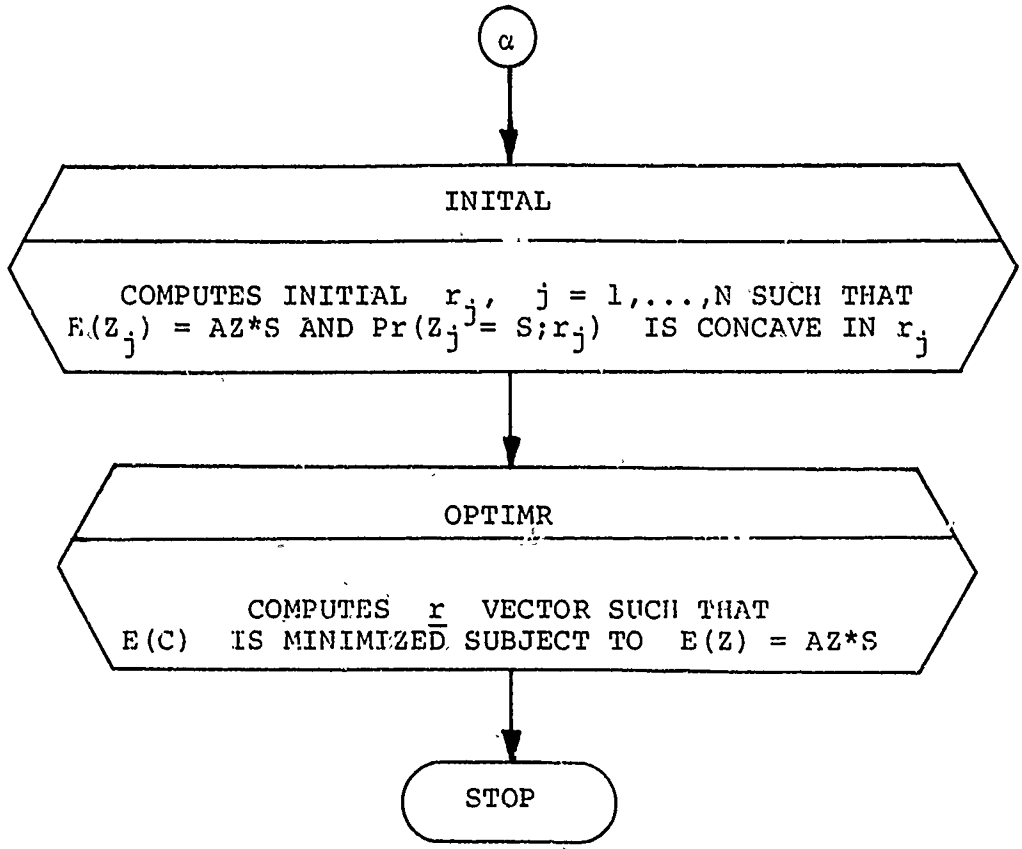


Figure B1. Flow Chart for EZMIN Main Program


α
INITAL
COMPUTES INITIAL r_j, $j = 1,\ldots,N$ SUCH THAT $E(Z_j) = AZ*S$ AND $Pr(Z_j = S; r_j)$ IS CONCAVE IN r_j
OPTIMR
COMPUTES $\underline{r}$ VECTOR SUCH THAT $E(C)$ IS MINIMIZED SUBJECT TO $E(Z) = AZ*S$
STOP

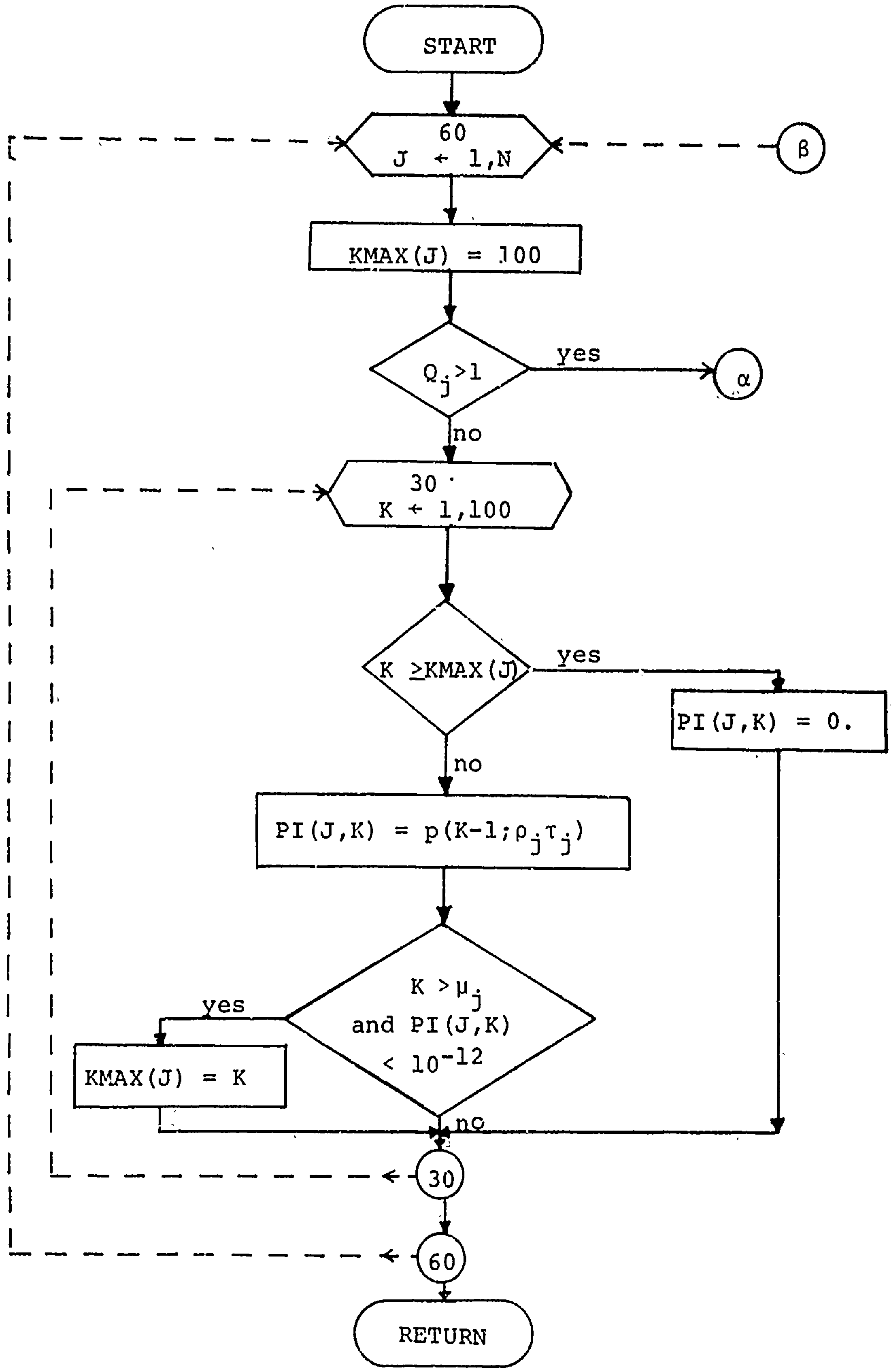


Figure B2. Flow Chart for Subroutine PSITAB

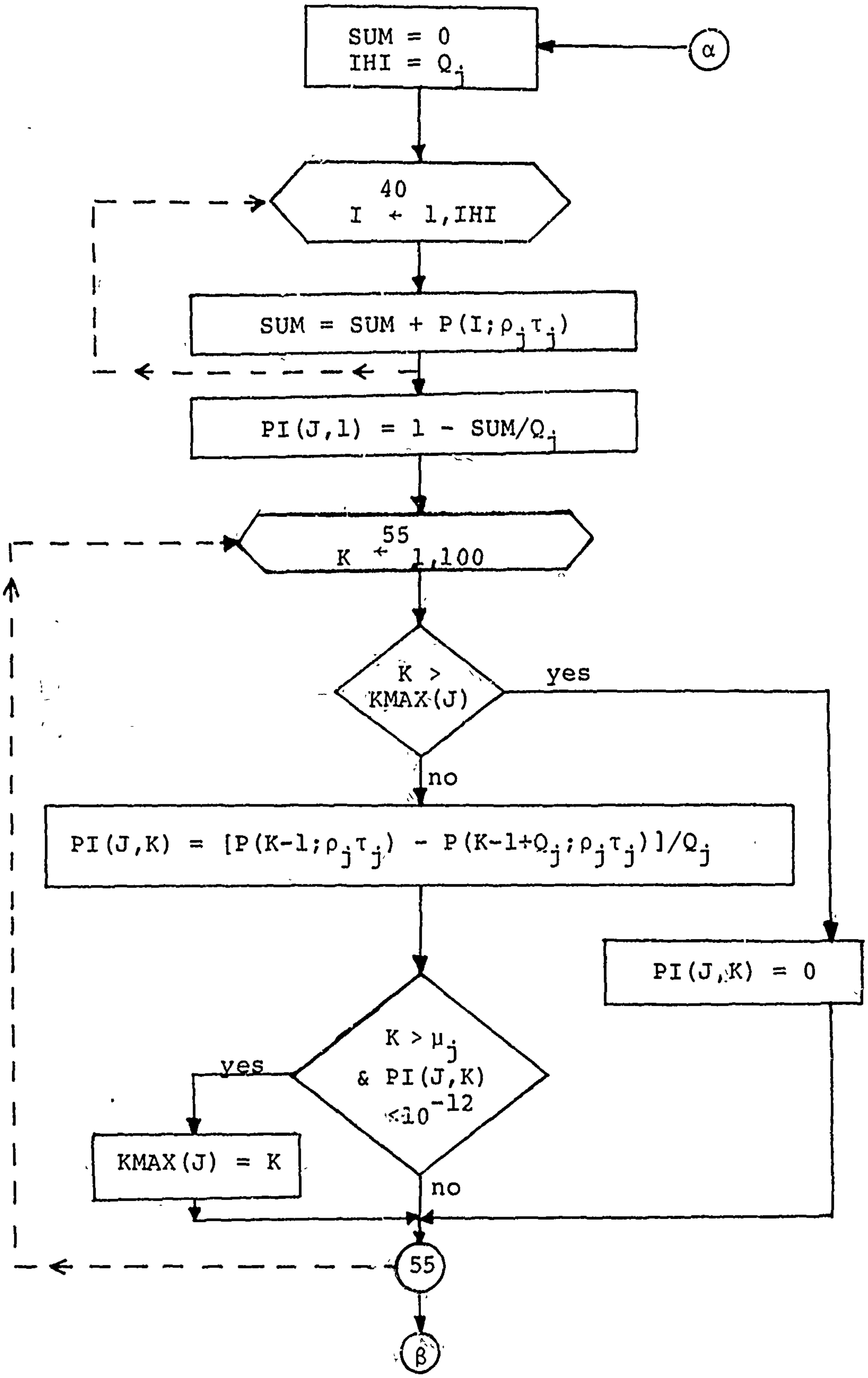


Figure B2. Continued

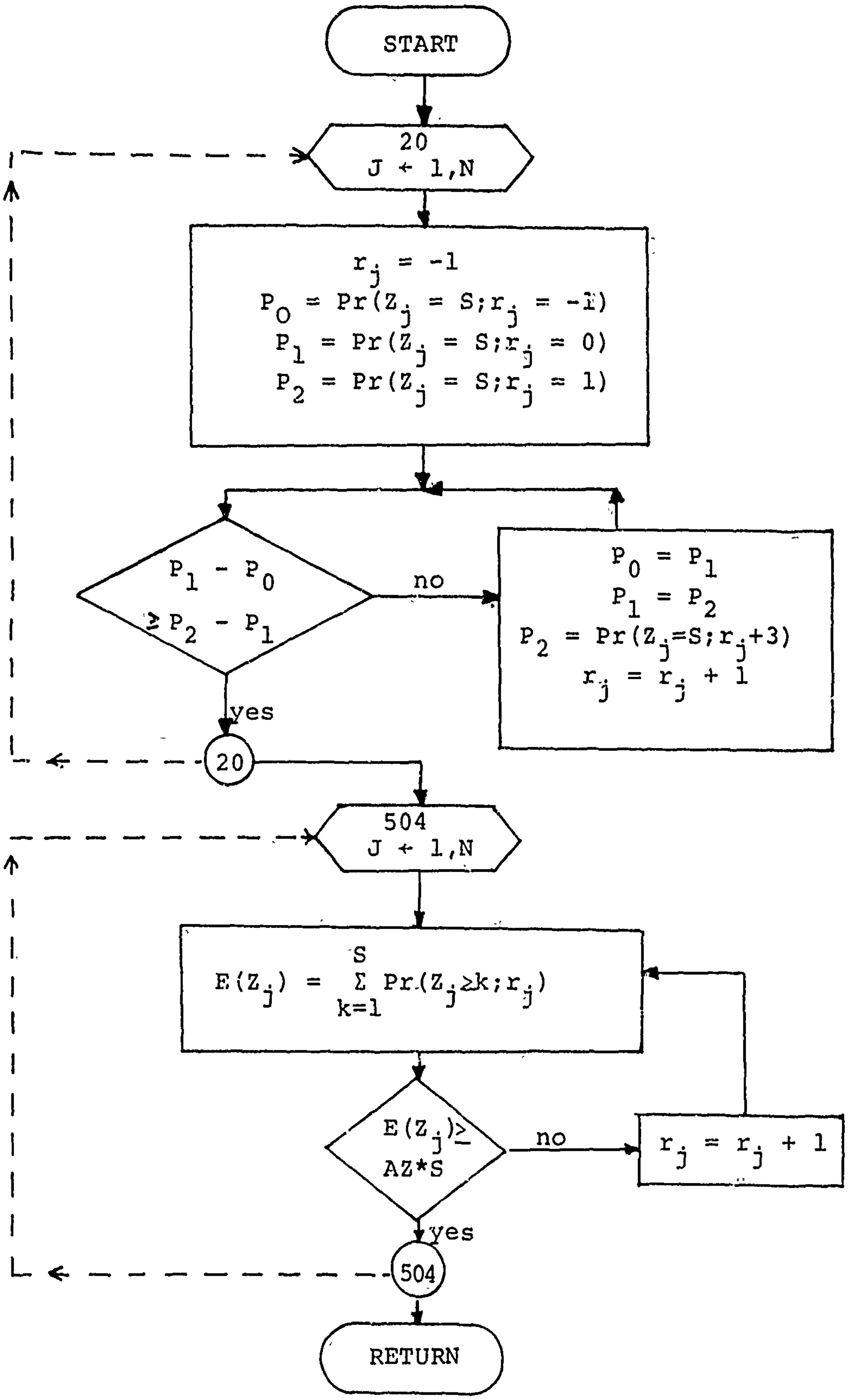


Figure B3. Flow Chart for Subroutine INITAL

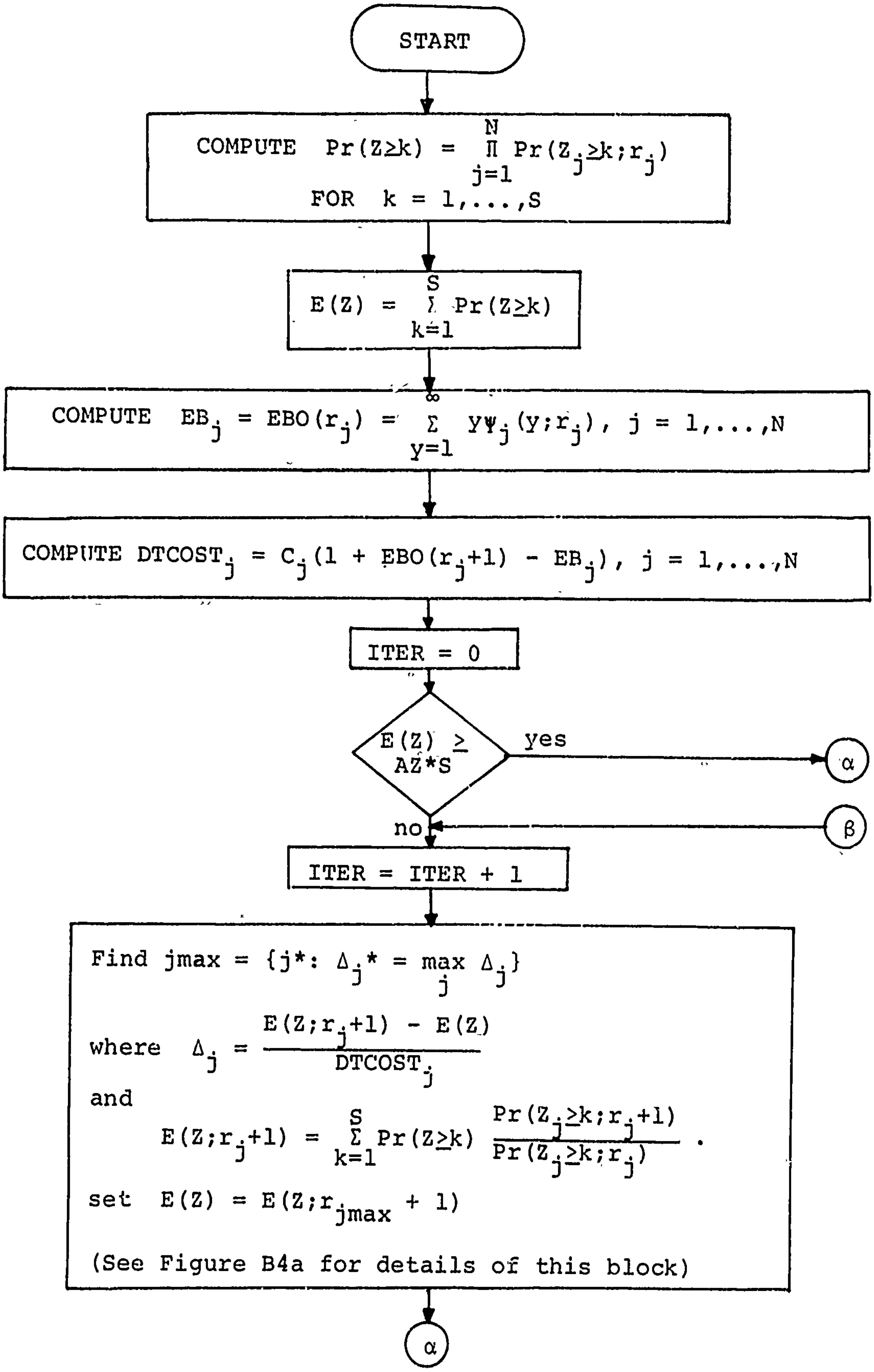


Figure B4. Flow Chart for Subroutine OPTIMR

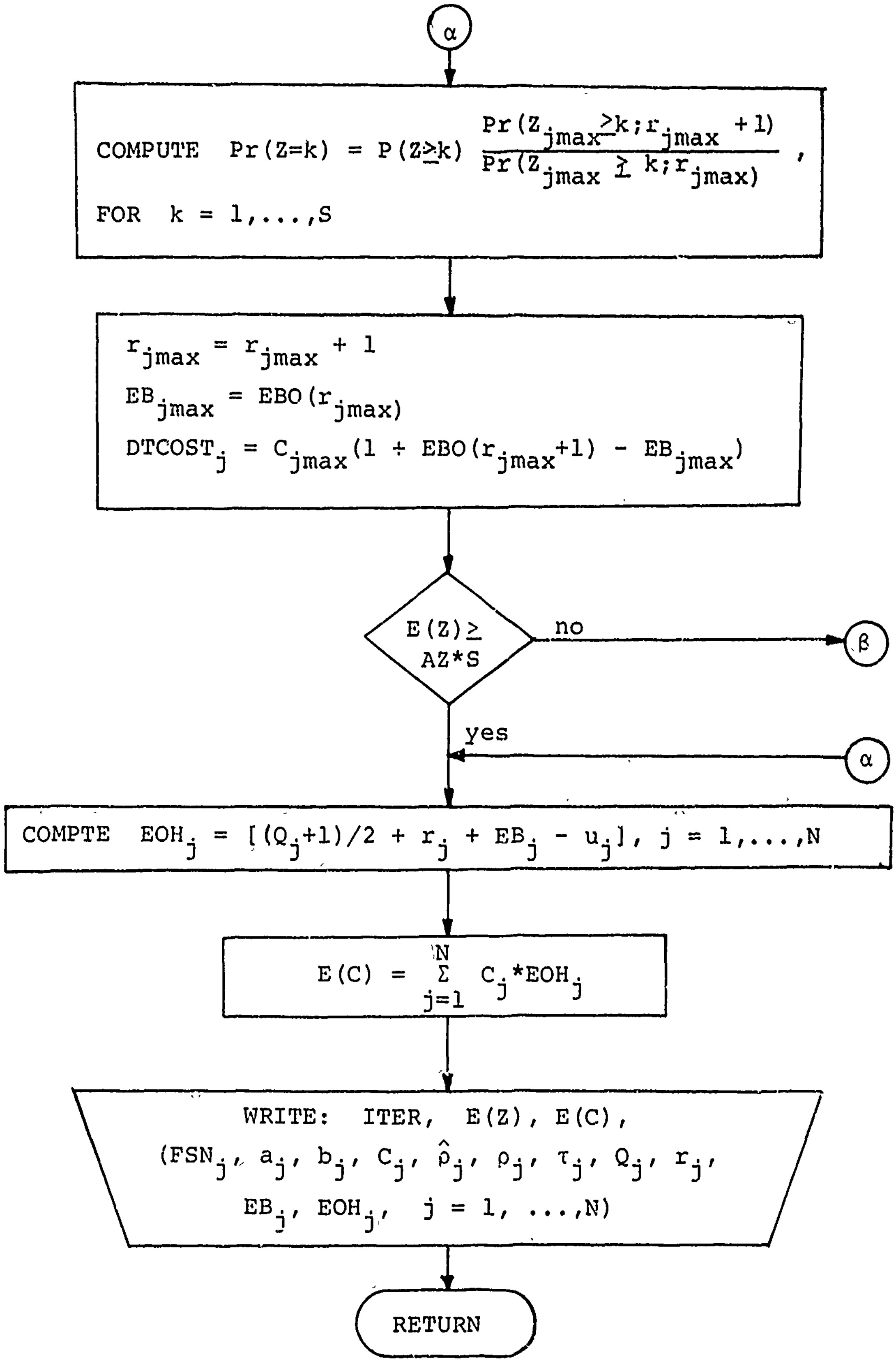


Figure B4. Continued

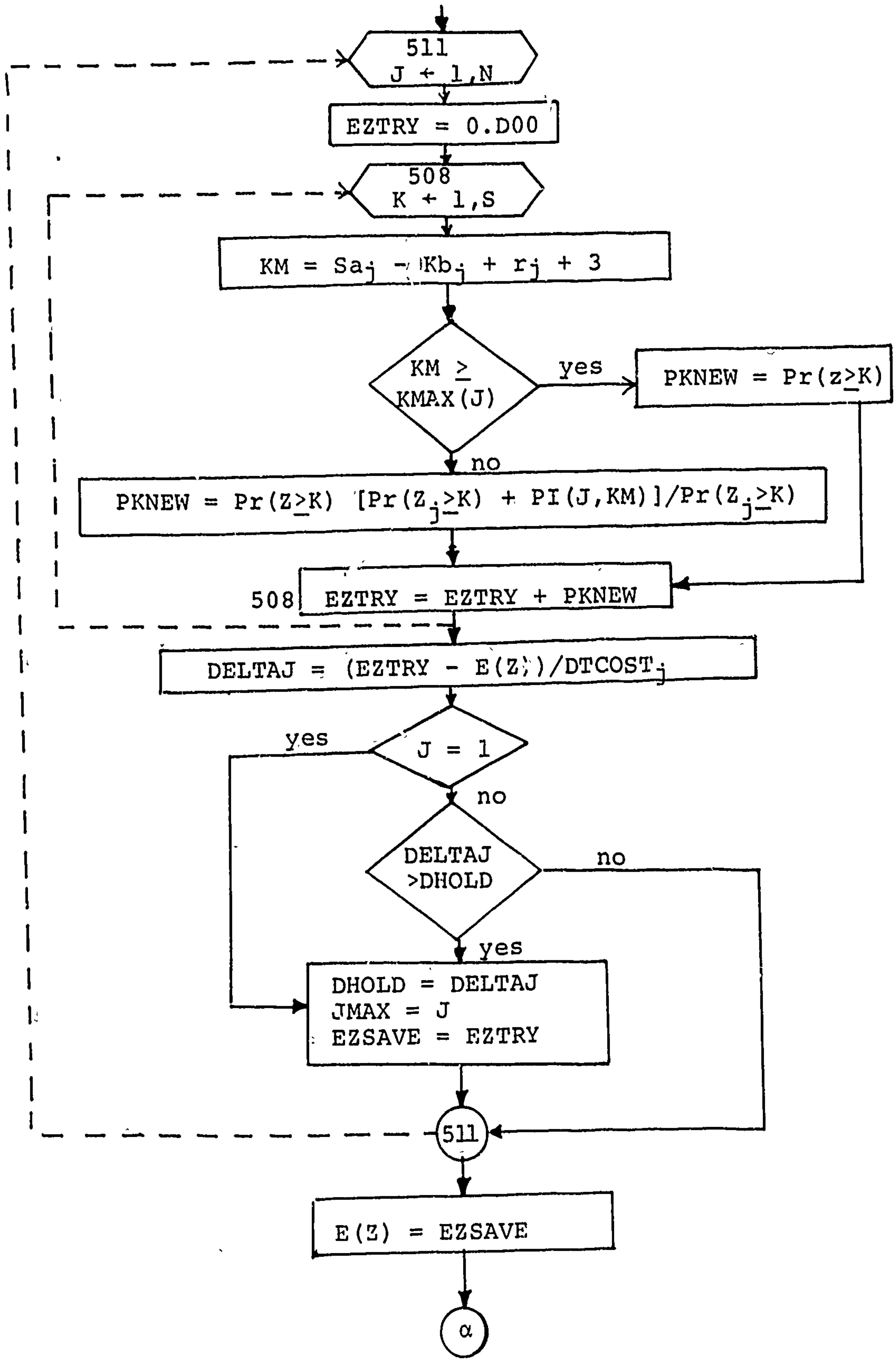


Figure B4a. Flow Chart for JMAX Segment of OPTIMR

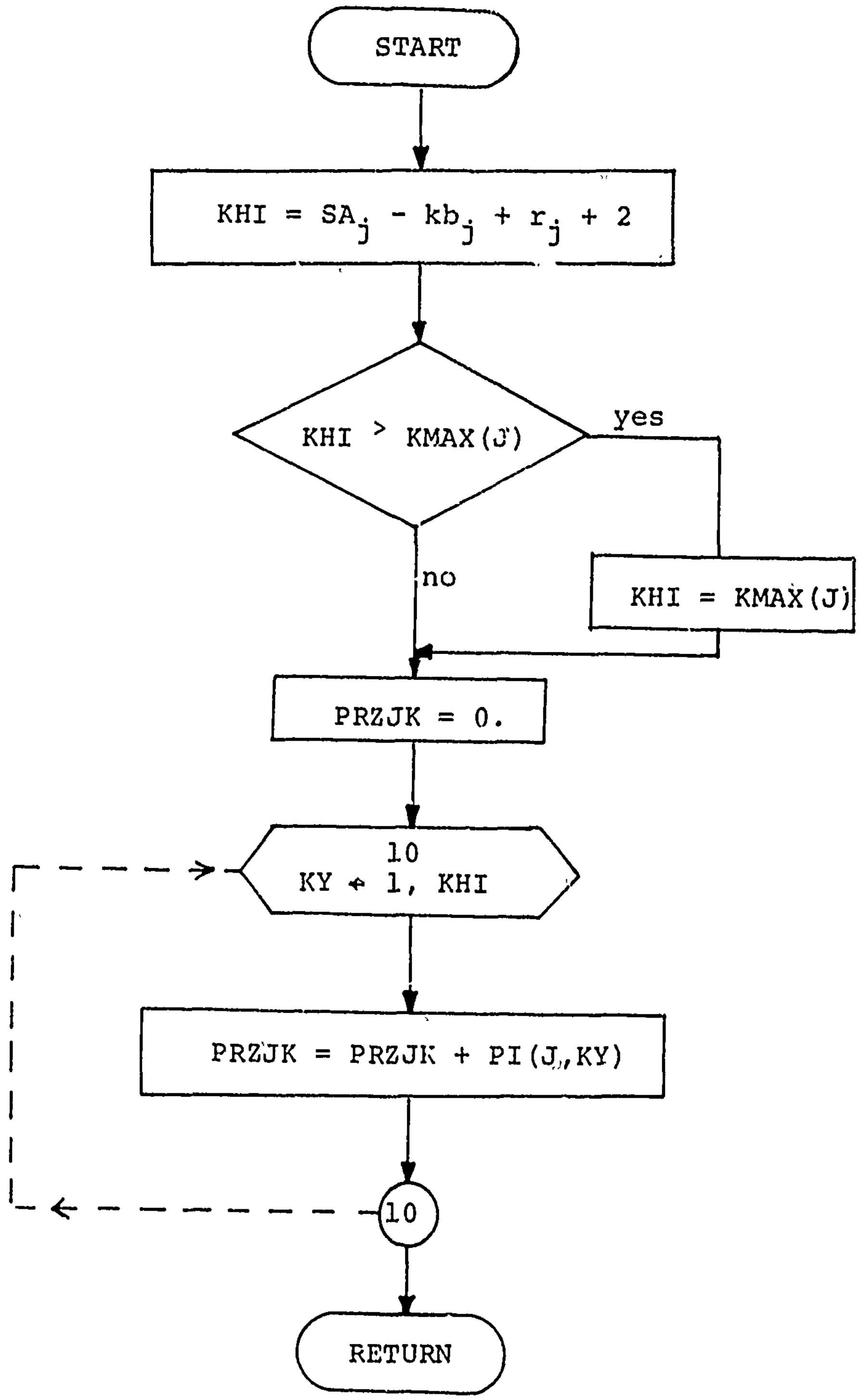


Figure B5. Flow Chart for Function PRZJK

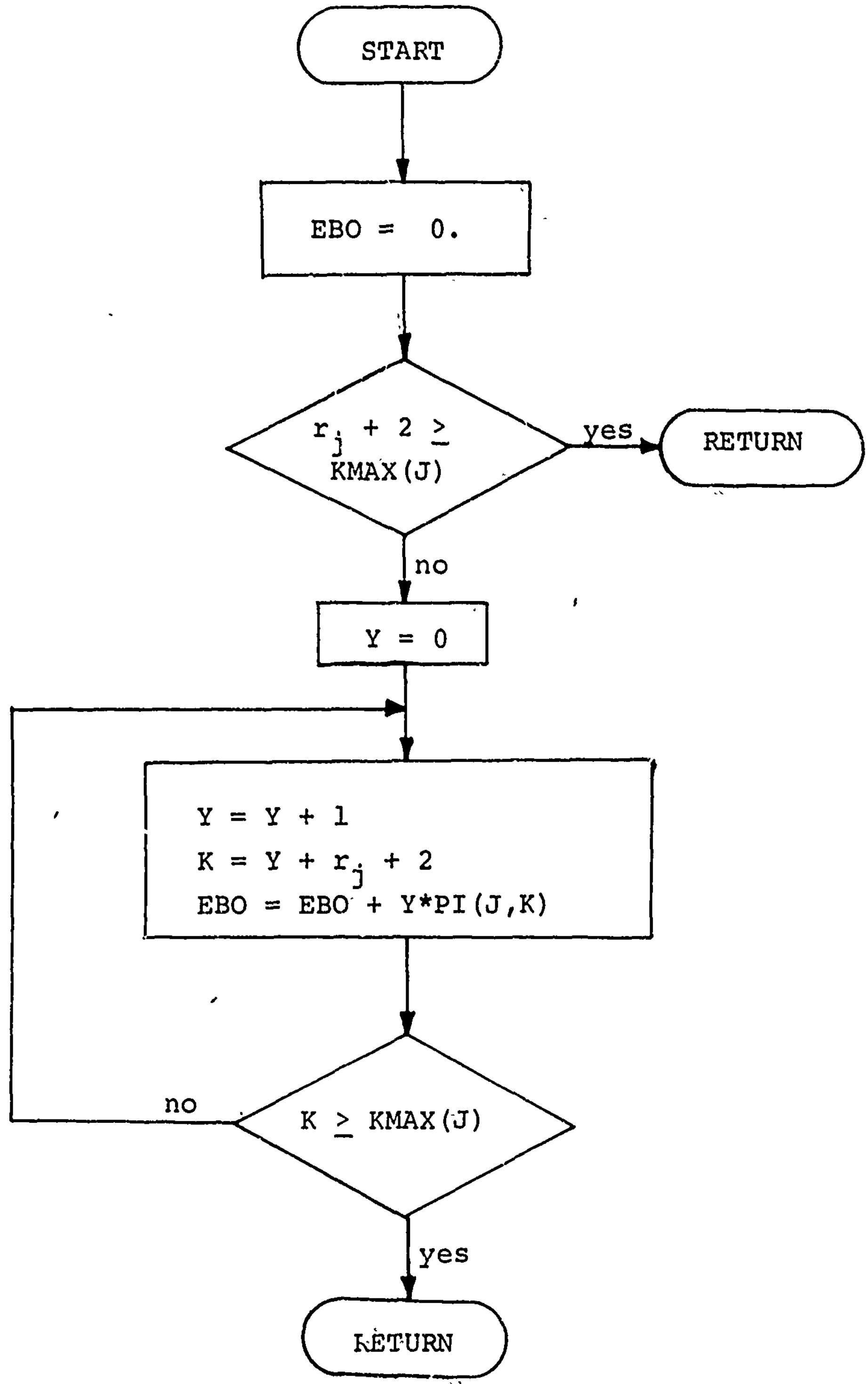


Figure B6. Flow Chart for Function EBO

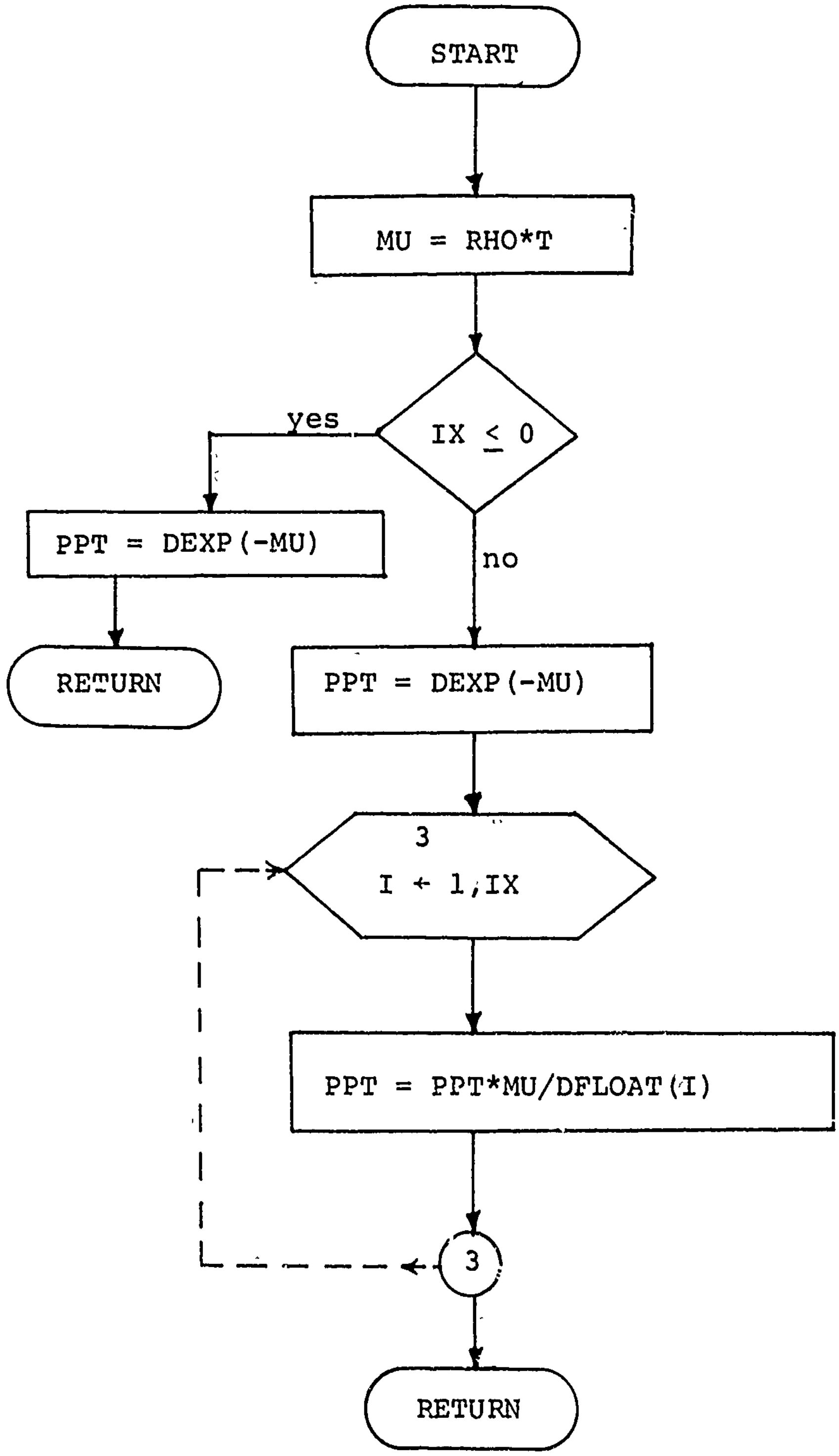


Figure B7. Flow Chart for Function PPT

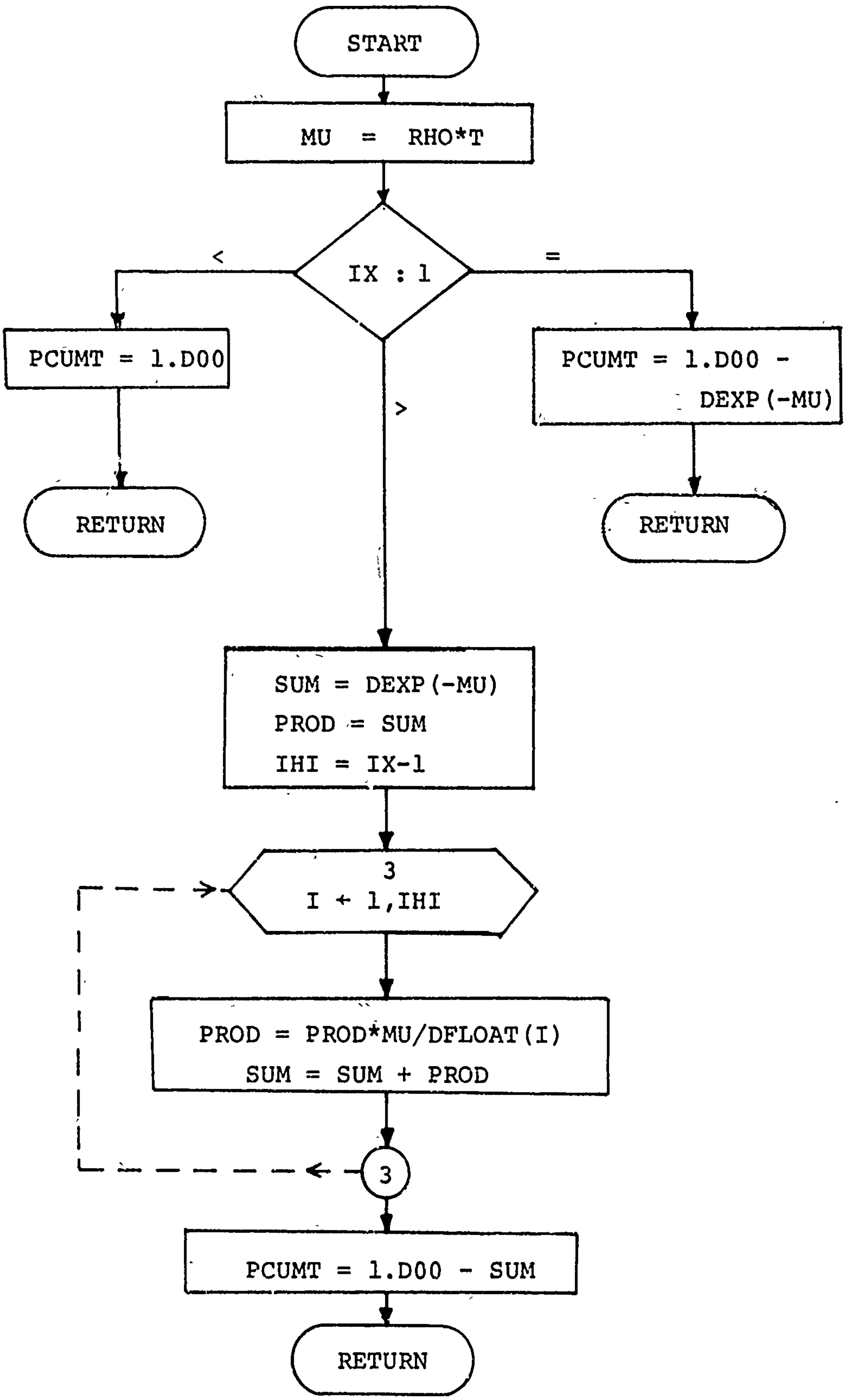


Figure B8. Flow Chart for Function PCUMT

APPENDIX C.

FLOW CHARTS FOR FORTRAN PROGRAM PKSMIN

FORTRAN program PKSMIN was developed to solve the problem of finding the $\underline{r}$ vector which minimizes the expected cost of on hand inventory subject to a required probability that at least k unit systems are up. This appendix contains flow charts for the main program and the following subroutines:

Subroutine INTLZ, which compute a set of initial values of r_j such that $Pr(Z_j \geq k) \geq P(Z \geq k)_{min}$, $j=1,...,N$.

Subroutine OPTMZ, which uses a marginal analysis technique to find an optimal $\underline{r}$ vector.

Program PKSMIN also uses subroutine PSITAB and function subprograms PRZJK, EBO, PPT, and PCUMT, flow charts for which are in Appendix B.

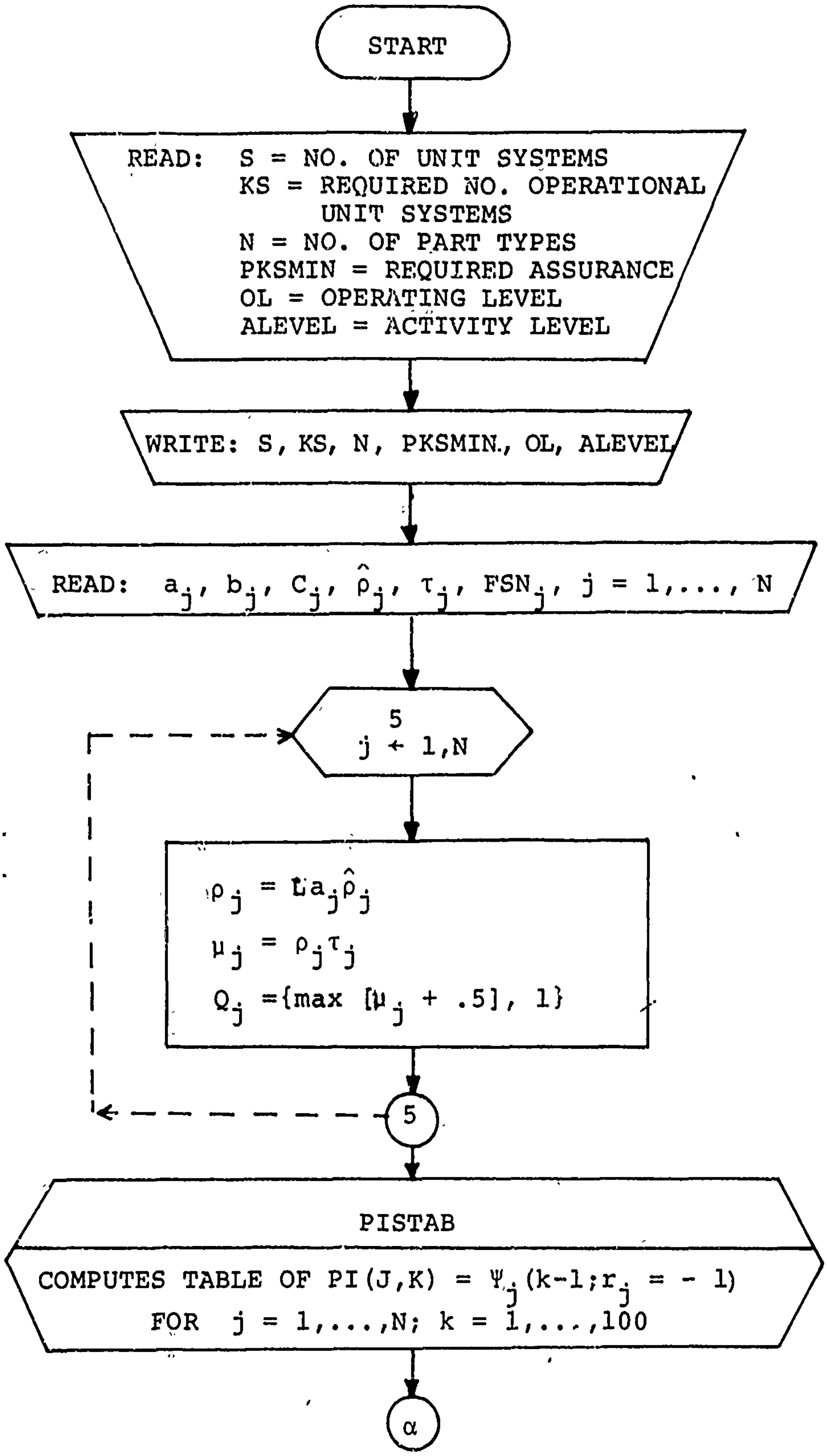


Figure C1. Flow Chart for PKSMIN Main Program

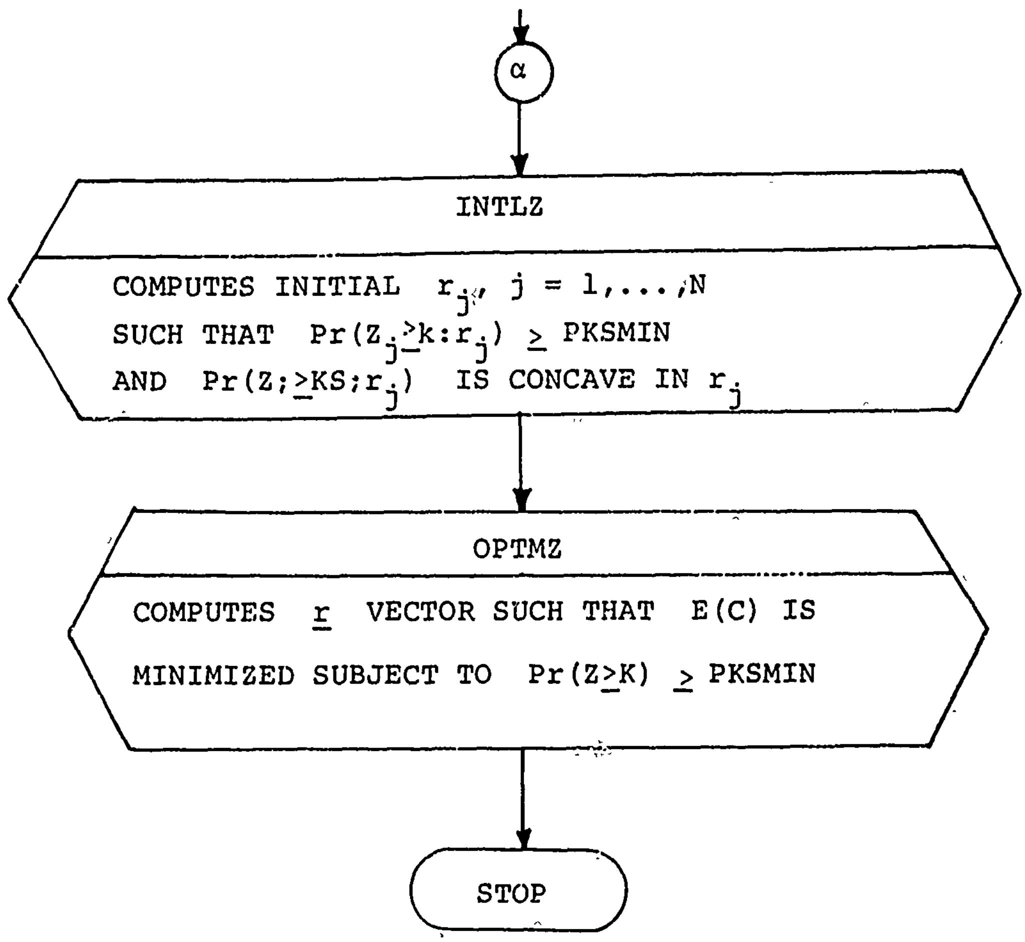


Figure C1. Continued

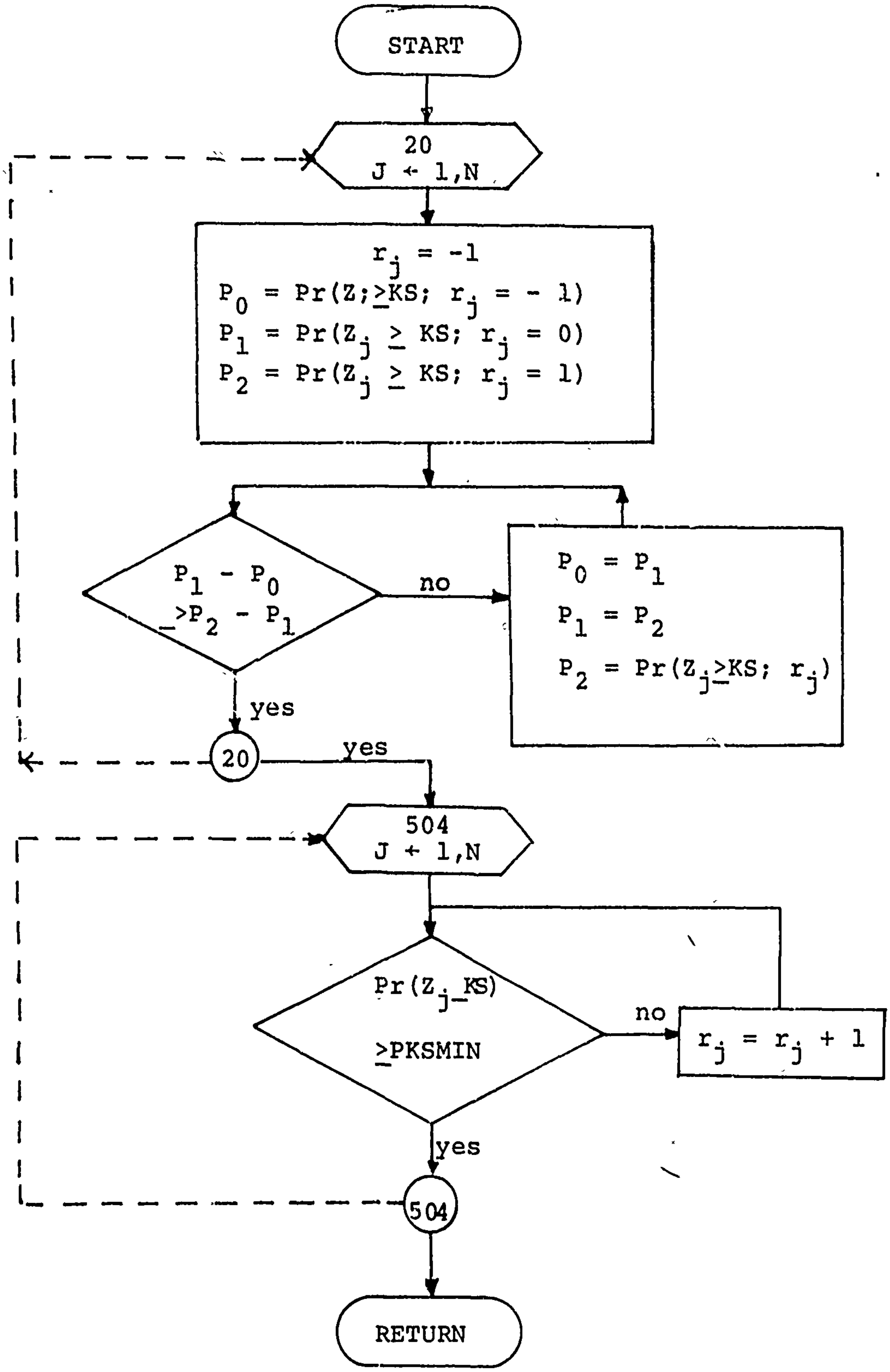


Figure C2. Flow Chart for Subroutine INTLZ

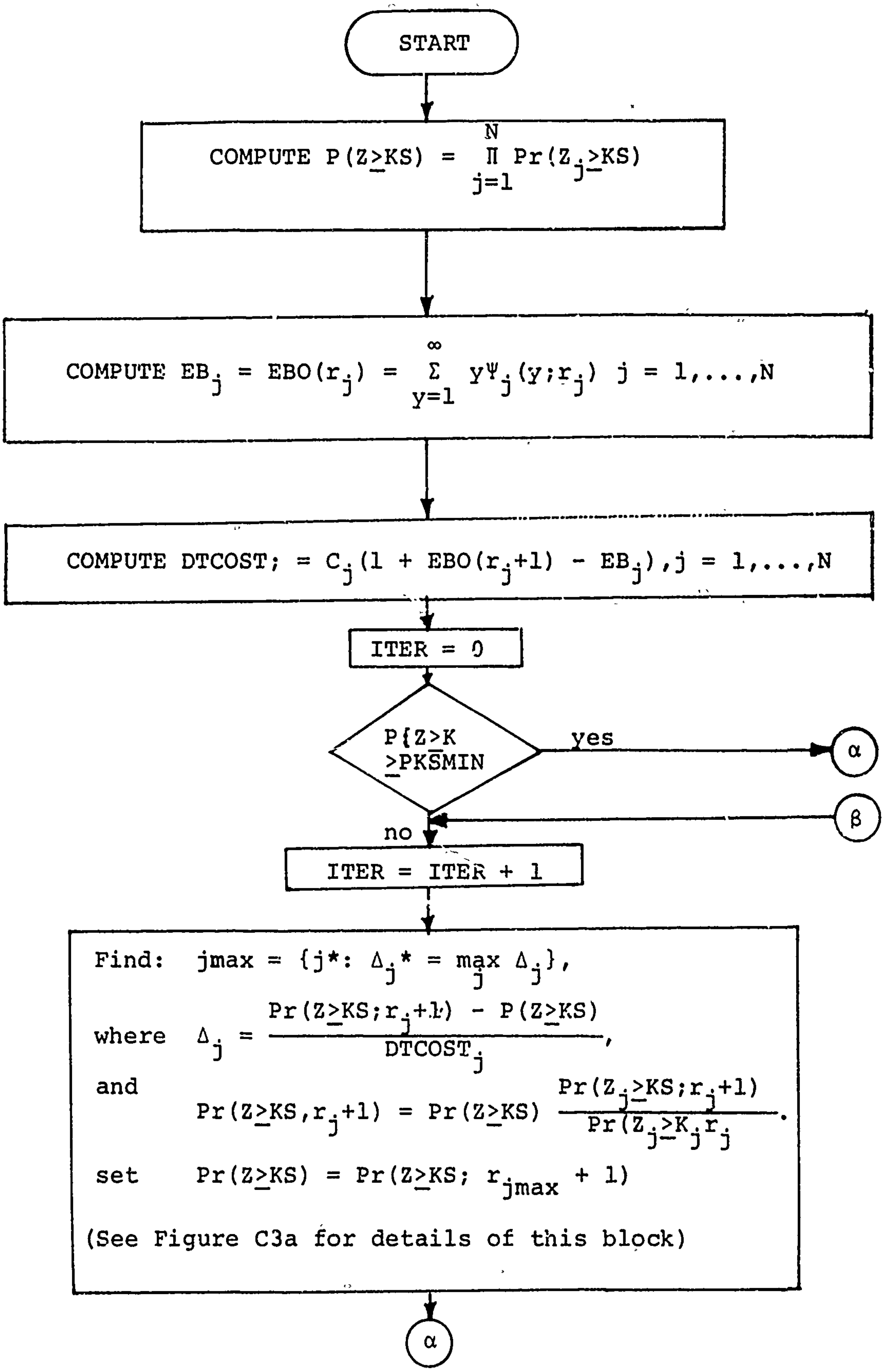


Figure C3. Flow Chart for Subroutine OPTMZ

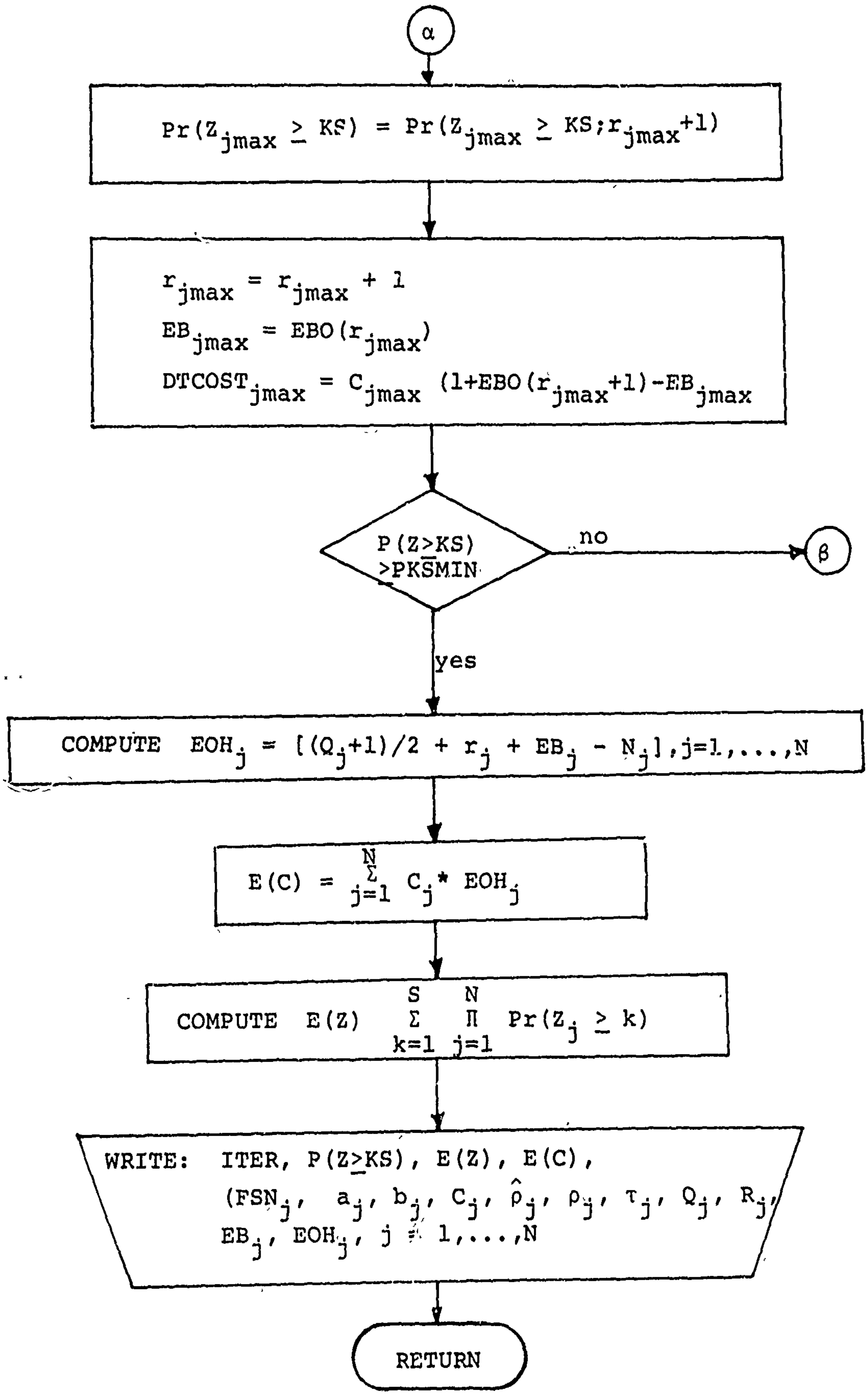


Figure C3. Continued

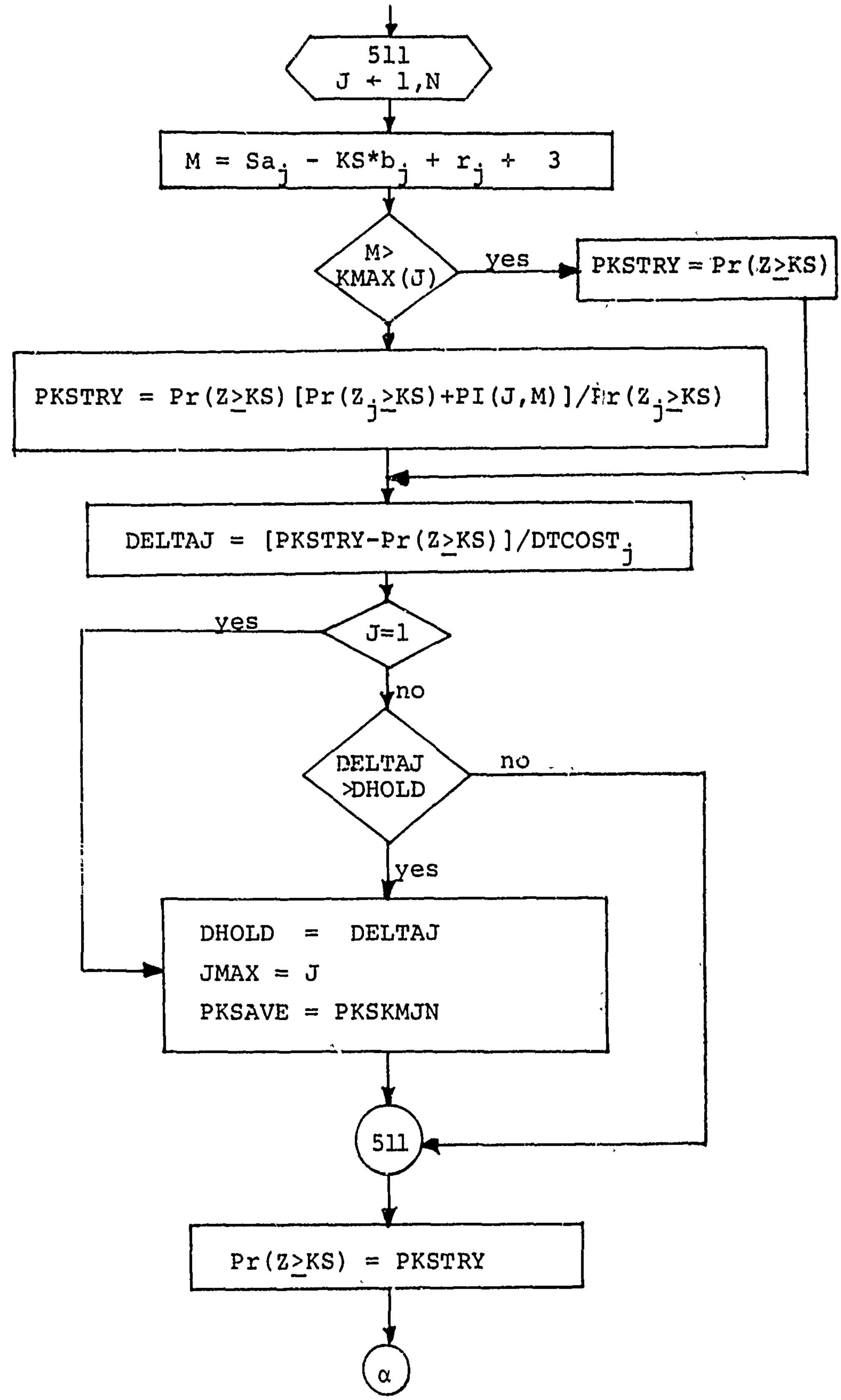


Figure C3a. Flow Chart for JAMX Segment of OPTMZ

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13. ABSTRACT

A model is developed for relating continuous review inventory policies for repair parts to system availability. The system consists of S identical unit systems, each of which is a series of k-out-of-n structures. Unit system states are zero or one. An optimal cannibalization policy is assumed. Under this assumption the number of unit systems up is always the maximum possible for any given vector of backorders for the N part types in the system. The distribution of backorders under a (Q,r) policy with Poisson demands for each part type is used to derive expressions for system availability as functions of the Q,r vectors. For simplicity it is assumed that order quantities are set by an operating level in terms of days of supply. A numerical technique is presented for finding the vector of reorder points (safety levels) which minimizes the expected cost of on hand inventory subject to one of two alternative availability constraints.

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OPTIMAL RESOURCE ALLOCATION						
REPAIR PARTS SUPPLY						
STOCHATIC MODELS						
CONTINUOUS REVIEW INVENTORY POLICIES						